A Nash-in-Nash model of corporate control and oligopolistic competition under common ownership*

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Abstract

This paper proposes a model of corporate control and oligopolistic competition under common ownership. Each firm's conduct results from Nash bargaining (NB) among shareholders and firms play a Nash equilibrium in Nash bargains. NB encompasses a rich class of models of corporate control under common ownership, including the current canonical model due to O'Brien and Salop (2000, OS), which has however important deficiencies. A specification of NB overcomes these deficiencies and yields theoretical results and policy implications that contradict those derived under OS. I use Nash-in-Nash to study the competitive effects of changes in corporate control providing a rationale for a policy proposal requiring institutional investors to be passive.

Keywords: common ownership, corporate control, bargaining, Nash bargaining, Nash-in-Nash, minority shareholdings, antitrust policy, competition policy, oligopoly
JEL classification codes: C71, D43, G34, L11, L13, L21, L41

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1 Introduction

Perfect competition is crucial for shareholders to unanimously agree on own-firm profit maximization (Hart, 1979), which has been the standard assumption on firm behavior at least since Fisher's (1930) separation theorem. Yet, recent work has shown that firm market power has been widespread and increasing in the U.S. economy (Loecker et al., 2020). At the same time, there is evidence that absent perfect competition firms may not seek to maximize own profit. Particularly, common ownership has been argued to induce firms to (partially) internalize the effect their actions have on competing firms' profits, thus softening competition (e.g., see Posner et al., 2017; Azar et al., 2018; Schmalz, 2018).

In studying such anti-competitive effects, a model of corporate control other than own-profit maximization is necessary. This model will translate an industry's ownership structure into corporate conduct, which will in turn translate into equilibrium outcomes. The model needs to describe how firm policy will be shaped from shareholders' conflicting interests. For example, shareholders with smaller holdings in competing firms will want the firm to price more aggressively than shareholders with larger stakes in other firms.

Modeling corporate control can be more or less complicated depending on the ownership structure. When a common owner (*i.e.*, an investor that holds shares in multiple firms within an industry) holds the majority of a firm's shares, it is natural to model that firm as trying to maximize that common owner's wealth (from her holdings across all firms). Similarly, if there is a majority non-common owner, then it is reasonable to assume that the firm will seek to maximize its own profit.[1] However, in practice most large firms are held by multiple minority shareholders, whose holdings across firms in an industry vary. It is then not as simple to decide on a satisfying model of corporate control.

Existing work has so far followed O'Brien and Salop (2000) in using the following model of corporate control, which I call the weighted average portfolio profit (WAPP) model.[2] Given a set N of investors who hold all shares in the industry and a set M of firms, each firm j maximizes a weighted average of its shareholders' portfolio profits, that

[1] Previous works have recognized this and for simplicity assumed each firm to be controlled by a majority shareholder (e.g., see Anton et al., 2021).

[2] Similar ideas can be traced back to Rotemberg (1984) and Bresnahan and Salop (1986).

is

$$\sum_{i\in N}\gamma_{ij}\sum_{k\in M}s_{ik}\pi_k \propto \pi_j + \sum_{k\in M\setminus\{j\}} \overbrace{\frac{\sum_{i\in N}\gamma_{ij}s_{ik}}{\sum_{i\in N}\gamma_{ij}s_{ij}}}^{=:\lambda_{jk}} \pi_k,$$

where s_{ik} is shareholder i's cash-flow right over firm k's profits, γ_{ik} her control power over firm k (which depends on the ownership structure s_{*k} of the firm), and π_k is firm k's profit.[3] That is, firm j's manager maximizes a weighted average of firm j's shareholders' portfolio profits (*i.e.*, their total earnings from their cash-flow rights over all firms in the industry). Equivalently, firm j maximizes its own profit plus each other firm k's profit weighted by λ_{jk}, the Edgeworth coefficient of effective sympathy from firm j towards firm k (*i.e.*, the weight firm j assigns to firm k's profit with the weight on its own profit normalized to 1). It is said that there is proportional control if $\gamma_{ik} = s_{ik}$ for all i and k.

However, the WAPP model has some undesirable properties. It has been criticized to make counter-intuitive predictions regarding the effect of ownership dispersion within a firm j on the degree λ_{jk} to which that firm will internalize another firm k's profit (e.g., see Gramlich and Grundl, 2017; O'Brien and Waehrer, 2017; Brito et al., 2023). As ownership by non-common (resp. common) owners becomes dispersed, firm j tends to follow only the common (resp. non-common) owners' interests.[4] While to an extent this effect seems reasonable, the WAPP model mechanically produces it to an extreme degree for almost any assumption on γ's (including proportional γ's). The effect is particularly important given that large investment funds are more diversified than smaller shareholders.

I propose an alternative model of corporate control, where firm j's conduct is modeled as the result of (asymmetric) Nash bargaining (NB) among firm j's shareholders with the actions of the other firms taken as given. Shareholders bargain á la Nash given a disagreement payoff function that maps each possible ownership structure in the industry and action profile of the other firms to a shareholder payoff vector (that will result from the action that firm j will take in case of disagreement among its shareholders). The

[3] If all shares have cash-flow rights, s_{ik} is equal to the proportion of firm k's shares that are held by i.

[4] For example, consider the case where firm j has two types of shareholders: (i) common owners, who are invested equally in each firm in the industry, and (ii) non-common owners, who only hold shares of firm j. Under any realistic assumptions on γ's, if we hold fixed the total amount of shares held by each group of shareholders, as shares held by non-common owners are dispersed across more non-common owners, λ_{jk} increases and in the limit where the number of non-common owners goes to infinity, $\lambda_{jk} \to 1$. The case where the number of common owners increases is analogous.

weights (*i.e.*, exponents) in the asymmetric Nash product of firm j are a function of firm j's ownership structure s_{*j}, as is γ_{*j} in WAPP. The equilibrium concept is then a Nash equilibrium in Nash bargains.

Like WAPP, the NB model can best be understood as an "as-if" assumption. This "as-if" approach allows for a rich generalization of the WAPP model. There are five main takeaways.

First, using the NB model is without loss of generality in the following sense. Restricting attention to NB mechanisms of corporate control essentially amounts to only considering the class of efficient mechanisms (*i.e.*, mechanisms that never lead to firm actions such that alternative actions by the firm would make every controlling shareholder of that firm weakly better off and at least one strictly so).[5] Although appealing, efficiency is a minimal assumption. Thus, obtaining a satisfying model of corporate control amounts to choosing disagreement payoff and weights functions in the Nash product that will give rise to additional (to efficiency) desirable properties. The WAPP mechanism is efficient and thus a specific way of choosing these functions but with the shortcomings described above.

Second, the random dictatorship (RD) specification of the disagreement payoff function solves this problem. This specification poses that in case of disagreement among the shareholders of firm j, a lottery is conducted: each shareholder gets to pick with some exogenous probability the action of firm j and the disagreement payoff of each shareholder is her expected portfolio profit from this lottery.[6] Under this specification of NB, a natural connection between parameters in the Nash product and properties of a firm's best response function arises. Particularly, a proportional control assumption on the parameters of the model corresponds to a behavioral definition of proportional control (*i.e.*, a definition that refers directly to a firm's best response function rather than parameters of the functional form of its objective function).

Third, using NB with RD disagreement payoffs can deliver significantly different predictions and policy implications. I study the effect of a policy that restricts the level of common ownership in a Cournot duopoly and show that the policy increases consumer welfare under WAPP but may harm it under NB.

[5]This is a generalization of Pareto efficiency in the sense that Pareto efficiency need only be satisfied with regard to a (possibly strict) subset of the firm's shareholders, who I call "controlling" shareholders.

[6]The probabilities in the lottery depend on the ownership structure of the firm.

Fourth, the first order conditions that describe firm behavior under the NB model are analogous to those derived under the WAPP model.[7] Thus, given the generality of NB, it is sufficient to perform comparative statics exercises under NB; different specifications of NB—including WAPP—will then amount to using the corresponding control weights.

Last, I characterize the equilibrium of a homogeneous product Cournot market with common ownership under NB.[8] I show that if a firm is underproducing (resp. overproducing) relative to a shareholder's preferences and that shareholder's control over that firm increases exogenously, then that firm's quantity will increase (resp. decrease). In the standard case, this means that the price will increase (resp. decrease). Particularly, I derive an intuitive measure of whether the firm is under- or overproducing relative to the investor's preferences providing a rationale for the policy proposal by Posner et al. (2017) that institutional investors be required to be passive if they accumulate large amounts of shares in multiple competing firms.

After this introduction, section 2 reviews related literature. Section 3 presents the model and section 4 provides a characterization of NB and compares it to WAPP. Section 5 applies NB to a homogeneous product Cournot market. Section 6 concludes. All proofs are gathered in Appendix A. Appendix B provides supplementary results.

2 Related literature

The Nash-in-Nash solution concept has become a standard tool, since it was proposed by Horn and Wolinsky (1988), who study merger incentives when there are exclusive vertical relationships. The current paper fits into the wide literature that has leveraged the Nash-in-Nash solution to study equilibrium outcomes in various environments where the division of surplus between parties (e.g., upstream and downstream firms) plays an important role.[9] It applies it to the case of oligopolistic competition among firms when within each firm, shareholders (with varying levels of holdings in competing firms) bargain to decide on firm policy.

[7]The only difference is the following. Under WAPP, the exogenous control weights of firm j's shareholders, γ_{*j} (depending only on the exogenous ownership structure s_{*j} of the firm), appear in the FOC. Under NB, analogous control weights appear in the FOC but are endogenously determined. Therefore, equilibria under NB and WAPP have analogous characterizations. Of course, the endogeneity of control weights comes with a loss in tractability.

[8]Given the above, this characterization is also valid under WAPP.

[9]For a review of related literature see Collard-Wexler et al. (2019), who also offer a non-cooperative foundation for the solution concept for the case of multiple upstream and downstream firms.

In contrast, theoretical work on corporate control under common ownership has so far focused on microfounding the WAPP mechanism in models of shareholder voting (e.g., see Azar, 2017; Brito et al., 2018; Moskalev, 2019). Azar and Ribeiro (2022) go a step further modifying the voting model to account for managerial entrenchment, which leads to a generalization of WAPP.[10] They assume that the manager's preference is to maximize her firm's own profit, which implies that relative to WAPP their model is closer to own-profit maximization. Their model also predicts that as ownership becomes dispersed, the manager has more power and thus internalizes the shareholders' interests to a lesser degree. Although their empirical estimates are qualitatively consistent with this prediction, they show that their voting model overstates this effect. Crucially, it predicts that as ownership becomes infinitely dispersed, the manager tends (in the limit) to maximize own profit—even if all the firm's owners are completely diversified across the industry.

Brito et al. (2023) also try to overcome the shortcomings of WAPP by modifying some of the assumptions in the voting models that microfound WAPP. They argue that under certain assumptions, the resulting weighted average profit weight (WAPW) model does not give excessively more power to larger shareholders. However, I show that WAPW is a reframing of WAPP and even though it indeed gives rise to a parametrization of WAPP which deals with the issue, that parametrization is unrealistic: it gives all shareholders of a firm the same amount of control, so that the firm maximizes the *unweighted* average of its shareholders' portfolio profits.

Apart from overcoming these issues, my approach also differs methodologically from previous works. Instead of microfounding a corporate control model through shareholder voting, I take an axiomatic approach, which allows for more flexibility and avoids the narrow predictions of shareholder voting models. NB mechanisms are characterized as the class of efficient mechanisms and WAPP as a special case of NB. Proportional control is behaviorally defined in terms of a firm's best response correspondence and a natural connection between this definition and the parameters of the firm's objective function under NB is provided.

[10] It can be shown that their model is equivalent to a generalization of WAPP where the manager of firm j is treated as a "virtual" shareholder of the firm with control power γ_j^m and "cash-flow right" s_j^m normalized to $s_j^m = 1$ (so that $s_j^m + \sum_{i \in N} s_{ij} = 2$).

3 A model of corporate control and oligopolistic competition under common ownership

A tuple $G := \langle N, M, (A_j)_{j\in M}, (\pi_j)_{j\in M}, (s_{ij})_{(i,j)\in N\times M}\rangle$ characterizes an oligopoly game with common ownership, where $N := \{1,2,\ldots,n\}$ is a set of n investors, $M := \{1,2,\ldots,m\}$ is a set of m firms, A_j is firm j's action space. Let the action profile space be denoted by $A := \times_{j\in M} A_j$. For an action profile $a \equiv (a_1,\ldots,a_m) \in A$, where $a_j \in A_j$ is firm j's action, a_{-j} denotes the profile of actions of all firms except j, and accordingly $A_{-j} := \times_{k\neq j} A_k$. Firm j's profit function is $\pi_j : A \to \mathbb{R}$ and $s \in S := \left\{s \in [0,1]^n \times [0,1]^m : \sum_{i\in N} s_{ij} = 1\ \forall j \in M\right\}$ is the (exogenous) ownership matrix, where s_{ij} denotes investor i's share of firm j. This means that i has a cash-flow right over fraction s_{ij} of firm j's profits. Given a matrix A, let A_{i*} and A_{*j} denote A's i-th row and j-th column, respectively.[11]

Definition 1. An investor i is a shareholder of firm j if $s_{ij} > 0$. $N_j(s_{*j}) := \{i \in N : s_{ij} > 0\}$ is the set of shareholders of firm j.

Investor i's total portfolio profit function is $u_i(a,s_{i*}) := \sum_{j\in M} s_{ij}\pi_j(a)$. Define a mixed action profile $\alpha \equiv (\alpha_j)_{j\in M}$, where $\alpha_j \in \Delta(A_j)$ is a probability measure over A_j. $\pi_j(\alpha)$ is firm j's expected profit and $u_i(\alpha, s_{i*})$ is investor i's expected payoff from α.[12]

3.1 Individual firm behavior: corporate control mechanisms

A corporate control mechanism $g_j(\alpha_{-j},s)$ of firm j determines the set of actions deemed choosable by firm j for every ownership structure is s and the other firms' action profile α_{-j}.[13]

Definition 2. Firm j's corporate control mechanism is a correspondence $g_j : \times_{k\neq j}\Delta(A_{-k})\times S \rightrightarrows \Delta(A_j)$ that maps every ownership matrix $s \in S$ and action profile of the other firms $\alpha_{-j} \in \Delta(A_{-k})$ to a nonempty set $g_j(\alpha_{-j},s) \neq \emptyset$ of actions of firm j.

[11] The notation for a function that maps to an $n \times m$ space is analogous.

[12] Abusing notation I use both pure and mixed action profiles as arguments in the same function.

[13] In principle, the corporate control mechanism should describe firm behavior for any possible profit functions π, given that market conditions such as technology and demand may change. To economize on notation, I suppress this dependence on π.

3.1.1 The weighted average portfolio profit (WAPP) mechanism

Let Δ^n denote the n-dimensional simplex. I first describe the mechanism of O'Brien and Salop (2000), which I call the weighted average portfolio profit (WAPP) mechanism.

Definition 3. Firm j's corporate control mechanism g_j is a weighted average portfolio profit mechanism if there exists a control power function $\gamma_{*j} : \Delta^n \to \Delta^n$ such that for every $s \in S$ and $\alpha_{-j} \in \times_{k \neq j} \Delta(A_{-k})$

(i) (*weighted average portfolio profit maximization*)

$$g_j\left(\alpha_{-j}, s\right) = \mathrm{WAPP}_{\gamma_{*j}}\left(\alpha_{-j}, s\right) := \underset{\alpha_j \in \Delta(A_j)}{\arg\max} \left\{ \sum_{i \in N} \gamma_{ij}(s_{*j}) u_i(\alpha_j, \alpha_{-j}, s_{i*}) \right\},$$

(ii) (*control exclusive to shareholders*) for every $i \in N$, $s_{ij} = 0 \implies \gamma_{ij}(s_{*j}) = 0$.

This can be rewritten as

$$\mathrm{WAPP}_{\gamma_{*j}}\left(\alpha_{-j}, s\right) = \underset{\alpha_j \in \Delta(A_j)}{\arg\max} \left\{ \pi_j(\alpha) + \sum_{k \in M \setminus \{j\}} \overbrace{\frac{\sum_{i \in N} \gamma_{ij}(s_{*j}) s_{ik}}{\sum_{i \in N} \gamma_{ij}(s_{*j}) s_{ij}}}^{=:\lambda_{jk}(s) \geq 0} \pi_k(\alpha) \right\},$$

where $\lambda_{j*}(s)$ is the vector of weights firm j places in firms' profits with λ_{jj} normalized to 1. λ_{jk} is called the Edgeworth (1881) coefficient of effective sympathy of firm j towards firm k. The numerator of λ_{jk} is a measure of the level of cross-holdings of shareholders of firm j in firm $k \neq j$. The denominator measures ownership concentration in firm j.

In the literature, the case where the control weights γ_{*j} are proportional to the investors' shares in firm j is called proportional control. This is the most common assumption used in empirical work. An alternative specification that has received attention (e.g., see Azar and Vives, 2022) assumes γ_{*j} to be the (normalized) Banzhaf power indices of the shareholders.[14] Brito et al.'s (2023) model gives rise to an alternative function γ_{*j}, which I call modified Banzhaf. Finally, I describe a simple control power function that is a generalization of proportional control and can account for the case where large shareholders have more or less than proportional control.

[14]This index was studied by Penrose (1946), Banzhaf (1965) and Coleman (1971). To calculate the Banzhaf index, one first enumerates all winning (*i.e.*, with at least 50% of the firm's shares) coalitions of shareholders where there is (at least) one swing shareholder (*i.e.*, a shareholder who is in the coalition and by leaving it would make the coalition fail to reach majority). Then, the Banzhaf power index of a shareholder is the share of such coalitions where she is a swing shareholder.

Definition 4. The following control power functions will be used throughout the paper.

(i) The proportional control power function is $\gamma^P_{*j}(s_{*j}) := s_{*j}$ for every $s_{*j} \in \Delta^n$.

(ii) The Banzhaf control power function γ^B_{*j} specifies that for every $i \in N$ and $s_{*j} \in \Delta^n$

$$\gamma^B_{ij}(s_{*j}) := \frac{\left|\left\{T \in 2^N : \sum_{k\in T} s_{kj} \geq 1/2 > \sum_{k\in T\setminus\{i\}} s_{kj}\right\}\right|}{\sum_{t\in N}\left|\left\{T \in 2^N : \sum_{k\in T} s_{kj} \geq 1/2 > \sum_{k\in T\setminus\{t\}} s_{kj}\right\}\right|}.$$

(iii) The modified Banzhaf control power function γ^{mB}_{*j} specifies that for every $i \in N$ and $s_{*j} \in \Delta^n$

$$\gamma^{mB}_{ij}(s_{*j}) := \begin{cases} \frac{\gamma^B_{ij}(s_{*j})/s_{ij}}{\sum_{t\in N_j(\gamma^B_{*j})} \gamma^B_{tj}(s_{*j})/s_{tj}} & \text{if } \gamma^B_{ij}(s_{*j}) > 0 \\ 0 & \text{if } \gamma^B_{ij}(s_{*j}) = 0, \end{cases}$$

where $N_j(\gamma^B_{*j}) := \left\{i \in N : \gamma^B_{ij}(s_{*j}) > 0\right\}$.

(iv) For every $\theta \geq 0$, the single-parameter–θ control power function $\gamma^{sp-\theta}_{*j}$ specifies that for every $i \in N$ and $s_{*j} \in \Delta^n$

$$\gamma^{sp-\theta}_{ij}(s_{*j}) := \begin{cases} \frac{s^\theta_{ij}}{\sum_{t\in N_j(s_{*j})} s^\theta_{tj}} & \text{if } s_{ij} > 0 \\ 0 & \text{if } s_{ij} = 0. \end{cases}$$

For $\theta = 1$, $\gamma^{sp-1}_{*j} = \gamma^P_{*j}$. For $\theta = 0$, $\gamma^{sp-0}_{ij}(s_{*j}) = |N_j(s_{*j})|^{-1}$ for every shareholder i of firm j, that is, the firm maximizes the *unweighted* average of its shareholders' portfolio profits.

If all shares have voting rights, Brito et al.'s (2023) voting model gives rise to four specifications of γ_{*j} in the WAPP model, which are presented in Table 1. In their model, when the profit relevance of shareholder bias parameter is equal to 1, the authors frame the corporate control mechanism as—what I term—a weighted average profit weight (WAPW) mechanism.

To describe the WAPW mechanism, we first need to define the following.

Table 1: γ_{*j} under alternative assumptions in Brito et al. (2023)

Assumptions in Brito et al. (2023)	Profit relevance of shareholder bias in Brito et al. (2023)	γ_{*j}
1, 2, 4, 5 and 7	0	γ_{*j}^{P}
	1	γ_{*j}^{sp-0}
1, 3, 4, 5, 6 and 7	0	γ_{*j}^{B}
	1	γ_{*j}^{mB}

Definition 5. For every shareholder $i \in N_j(s_{*j})$ of firm j

$$\lambda_{i;j*} \equiv \begin{pmatrix} \lambda_{i;j1} & \lambda_{i;j2} & \cdots & \lambda_{i;jm} \end{pmatrix} := \frac{1}{s_{ij}} s_{i*} \equiv \begin{pmatrix} s_{i1}/s_{ij} & s_{i2}/s_{ij} & \cdots & s_{im}/s_{ij} \end{pmatrix}$$

is the vector of weights i places on firms' profits with the weight to firm j normalized to 1, where $\lambda_{i;jk} \equiv s_{ik}/s_{ij}$ is the weight she places on firm k's profit.

Definition 6. Firm j's corporate control mechanism g_j is a weighted average profit weight (WAPW) if there exists a control power function $\widehat{\gamma}_{*j} : \Delta^n \to \Delta^n$ such that for every $s \in S$ and $\alpha_{-j} \in \times_{k \neq j} \Delta(A_{-k})$

(i) (*weighted sum of firm profit maximization with weighted average profit weights*)

$$\begin{aligned} g_j\left(\alpha_{-j}, s\right) &= \mathrm{WAPW}_{\widehat{\gamma}_{*j}}\left(\alpha_{-j}, s\right) \\ &:= \underset{\alpha_j \in \Delta(A_j)}{\arg\max} \left\{ \pi_j(\alpha_j, \alpha_{-j}) + \sum_{k \in M \setminus \{j\}} \left(\sum_{i \in N_j(\gamma_{*j})} \widehat{\gamma}_{ij}(s_{*j}) \lambda_{i;jk} \right) \pi_k(\alpha_j, \alpha_{-j}) \right\}, \end{aligned}$$

where $N_j(\gamma_{*j}) \equiv \{i \in N : \widehat{\gamma}_{ij}(s_{*j}) > 0\}$,

(ii) (*control exclusive to shareholders*) for every $i \in N$, $s_{ij} = 0 \implies \widehat{\gamma}_{ij}(s_{*j}) = 0$.

In WAPW, the weight that the manager of firm j places on firm k's profit is a weighted average of the weights $\{\lambda_{i;jk}\}_{i \in N_j(s_{*j})}$ that the shareholders of firm j would want firm j to use. This still is a WAPP mechanism, since it can be written as

$$\mathrm{WAPW}_{\widehat{\gamma}_{*j}}\left(\alpha_{-j}, s\right) = \underset{\alpha_j \in \Delta(A_j)}{\arg\max} \left\{ \sum_{i \in N_j(\widehat{\gamma}_{*j})} \gamma_{ij}(s_{*j}) u_i(\alpha_j, \alpha_{-j}, s_{i*}) \right\},$$

where for every shareholder i of firm j

$$\gamma_{ij}(s_{*j}) \coloneqq \frac{\widehat{\gamma}_{ij}(s_{*j})/s_{ij}}{\sum_{i\in N_j(\widehat{\gamma}_{*j})} \widehat{\gamma}_{ij}(s_{*j})/s_{ij}}.$$

Thus, a mechanism is WAPP if and only if it is WAPW. The novelty is that the WAPW parametrizations considered in Brito et al. (2023) give rise to γ's that are not standard in the literature, namely γ^{sp-0}, which is derived from proportional $\widehat{\gamma}$'s, and γ^{mB}, which is derived from Banzhaf $\widehat{\gamma}$'s.

3.1.2 The Nash bargaining (NB) mechanism

I now describe the Nash bargaining (NB) corporate control mechanism.[15]

Definition 7. Firm j's corporate control mechanism g_j is a Nash bargaining mechanism if there exist a bargaining power function $\beta_{*j} : \Delta^n \to \Delta^n$ and a disagreement payoff function $d_{*j} : \times_{k\neq j}\Delta(A_k) \times S \to \mathbb{R}^n$ such that for every $s \in S$ and $\alpha_{-j} \in \times_{k\neq j}\Delta(A_{-k})$

(i) (*disagreement payoff feasibility*) there exists $\alpha_j \in \Delta(A_j)$ such that $d_{*j}(\alpha_j,\alpha_{-j},s) \leq u(\alpha_j,\alpha_{-j},s)$,[16]

(ii) (*Nash product maximization*)

$$\begin{aligned} g_j(\alpha_{-j},s) &= \mathrm{NB}_{\beta_{*j},d_{*j}}(\alpha_{-j},s) \\ &\coloneqq \underset{\alpha_j\in B_j(a_{-j},s)}{\arg\max} \left\{ \prod_{i\in N_j(\beta_{*j})} (u_i(\alpha_j,\alpha_{-j},s_{i*}) - d_{ij}(\alpha_{-j},s))^{\beta_{ij}(s_{*j})} \right\}, \end{aligned}$$

where $B_j(\alpha_{-j},s) \coloneqq \{\alpha_j \in \Delta(A_j) : u_i(\alpha_j,\alpha_{-j},s_{i*}) \geq d_{ij}(\alpha_{-j},s) \forall i \in N_j(\beta_{*j})\}$ and $N_j(\beta_{*j}) \equiv \{i \in N : \beta_{ij}(s_{*j}) > 0\}$,[17]

(iii) (*control exclusive to shareholders*) for every $i \in N$, $s_{ij} = 0 \implies \beta_{ij}(s_{*j}) = 0$.

Also, when firm j's mechanism is NB, an investor is called a controlling shareholder of firm j (at s_{*j}) if $\beta_{ij}(s_{*j}) > 0$.

[15] It is a maintained assumption that both mechanisms, WAPP and NB, are well-defined. Lemma 3 in the Appendix provides conditions for that to be the case under NB in a homogeneous product Cournot market.

[16] Given two vectors x,y, $x \geq y$ means $x_i \geq y_i$ for every i, $x > y$ means $x_i \geq y_i$ for every i with at least one inequality strict, while $x \gg y$ means $x_i > (<) y_i$ for every i; the relations $\leq, <, \ll$ are defined analogously.

[17] I write $N_j(\beta_{*j})$ instead of $N_j(\beta_{*j}(s_{*j}))$ to economize on notation.

Analogously to the WAPP mechanism, β_{*j} does not depend on π. However—although I suppress the dependence of mechanisms on profit functions to economize on notation—it is natural for d_{*j} to depend on π, as it refers to payoffs. The disagreement payoffs depend also on the other firms' actions, since firm j's course of action (with or without disagreement) will depend on the choices of the other firms.[18]

The random dictatorship disagreement payoff function I now propose the random dictatorship specification for disagreement payoffs, which poses that in case of disagreement in a firm, the shareholders' payoffs are derived from random dictatorship. With some exogenous probability each shareholder of the firm is chosen to implement her most preferred policy.

Definition 8. The disagreement payoff function d_{*j} is a random dictatorship (RD) disagreement payoff function if there exist a lottery weight function $\delta_{*j} : \Delta^n \to \Delta^n$ and a choice function (in case of disagreement) $\alpha_j^d : \times_{k \neq j} \Delta(A_k) \times \{v \in \mathbb{R}_+^m : v_j = 1\} \to \Delta(A_j)$ such that

(i) (*the choice function α_j^d for firm j is a selection from the correspondence that takes as arguments the other firms' actions α_{-j} and a vector v of relative weights on firms' profits (with the weight on firm j's profit normalized to* 1*) and returns the firm j action(s) that maximize the payoff of a shareholder with relative holdings v in the firms*) for every $v \in \{v' \in \mathbb{R}_+^m : v'_j = 1\}$[19]

$$\alpha_j^d(\alpha_{-j}, v) \in \arg\max_{\alpha_j \in \Delta(A_j)} \sum_{j \in M} v_j \pi_j(\alpha_j, \alpha_{-j}),$$

and for every $s \in S$ and $\alpha_{-j} \in \times_{k \neq j} \Delta(A_k)$

[18]For example, consider the following cases. If the other firms choose actions that drive the price lower than firm j's marginal cost, then there is essentially no disagreement and d_{*j} will reflect that the firm should not produce at all. On the other hand, if the other firms keep the price well above firm j's marginal cost, it is natural that in case of disagreement some level of production will take place in firm j.

[19]Notice that the choice function $\alpha_j^d(\alpha_{-j}, v_j)$ does not depend on the absolute size of a shareholder's stakes in the firms but only on her relative holdings v. This makes sense because $\arg\max_{\alpha_j \in \Delta(A_j)} \sum_{j \in M} v_j \pi_j(\alpha_j, \alpha_{-j})$ does not change if the objective function is multiplied by a positive constant. Also, notice that the choice function is not shareholder-specific. That is, all shareholders with the same relative holdings v choose the same action to be implemented by firm j in case of disagreement (if they are chosen by the lottery to make a decision). Of course, both of these properties are automatically satisfied when $\arg\max_{\alpha_j \in \Delta(A_j)} \sum_{j \in M} v_j \pi_j(\alpha_j, \alpha_{-j})$ is a singleton.

(ii) (*disagreement payoffs derived from random dictatorship*)

$$d_{*j}(\alpha_{-j},s) = \sum_{i\in N_j(\delta_{*j})} \delta_{ij}(s_{*j})u\left(\alpha_j^d\left(\alpha_{-j},\lambda_{i;j*}\right),\alpha_{-j},s\right),$$

where $N_j(\delta_{*j}) \equiv \{i \in N : \delta_{ij}(s_{*j}) > 0\}$ and $\lambda_{i;j*} \equiv s_{i*}/s_{ij}$,

(iii) (*control exclusive to shareholders*) for every $i \in N$, $s_{ij} = 0 \implies \delta_{ij}(s_{*j}) = 0$.

Definition 9. The proportional bargaining power and lottery weight functions are $\beta^P_{*j}(s_{*j}), \delta^P_{*j}(s_{*j}) := s_{*j}$.

RD disagreement payoffs have certain desirable properties. First, the disagreement payoffs are derived from a well-specified procedure. Second, they are feasible without the need for (free) disposal of profits. Third, through the probabilities δ with which different shareholders get to implement their most preferred action, the RD disagreement payoffs account for the relative power of shareholders.

Fourth, consider the case where A_j is a convex subset of a Euclidean space, and the portfolio profit of each firm j controlling shareholder is strictly concave in j's (pure) action a_j.[20] If firm j's controlling shareholders' preferences are not perfectly aligned,[21] then the shareholders have strict incentives to reach an agreement. Namely, by Jensen's inequality, every controlling shareholder will strictly prefer (to disagreement) that the firm implement the pure action that is the convex combination of the controlling shareholders' most-preferred actions,[22] so that the solution to the Nash bargaining problem is interior.

Last, while the NB mechanism can—much like the WAPP mechanism—be thought of as an as-if assumption, NB with RD disagreement payoffs (NBRD) also has connections to strategic foundations of Nash bargaining. For example, Howard (1992) shows that symmetric NBRD can be implemented as the unique perfect equilibrium outcome of a game.

[20]Lemma 2 in the Appendix provides sufficient conditions for strict concavity in a homogeneous product Cournot market.

[21]That is, there exist distinct $i,i' \in N$ such that $\delta_{ij}(s_{*j}),\delta_{i'j}(s_{*j}) > 0$ and $\alpha_j^d\left(\alpha_{-j},\lambda_{i;j*}\right) \neq \alpha_j^d\left(\alpha_{-j},\lambda_{i';j*}\right)$, which are singletons and pure actions by strict concavity. When firm j's controlling shareholders' preferences *are* perfectly aligned, then in case of disagreement, the action that is most preferred by all of them is chosen.

[22]That is, $u_i\left(\sum_{i\in N_j(\beta_{*j})}\delta_{ij}(s_{*j})\alpha_j^d\left(\alpha_{-j},\lambda_{i;j*}\right),\alpha_{-j}, s_{i*}\right) > d_i(\alpha_{-j},s)$ for every $i \in N_j(\beta_{*j})$.

3.2 Equilibrium definition and existence

I now present the equilibrium concept and prove the existence of a Nash equilibrium in Nash bargains.

Definition 10. Fix an $s \in S$. An action profile $\alpha \in \times_{j \in M} \Delta(A_j)$ is an equilibrium under corporate control mechanisms $(g_j)_{j \in M}$ if for every $j \in M$, $\alpha_j \in g_j(\alpha_{-j}, s)$.

Under NB, the equilibrium is a Nash equilibrium in Nash bargains. Particularly, the oligopoly game can be seen as a generalized game where a firm's action set depends on the other firms' actions. Namely, when the other firms play α_{-j}, firm j can choose an action in $B_j(\alpha_{-j}, s)$, because it needs to make sure that each controlling shareholder achieves at least her disagreement payoff. Proposition 1 provides sufficient conditions for existence of a pure equilibrium of this generalized game.

Proposition 1. Fix an $s \in S$. For each firm $j \in M$ let the corporate control mechanism g_j be $\mathrm{NB}_{\beta_{*j}, d_{*j}}$. If for every $j \in M$

(i) A_j is a non-empty, compact and convex subset of a Euclidean space,

(ii) $\pi_j(a)$ is continuous in a,

(iii) for each $i \in N$, $d_{ij}(a_{-j}, s)$ is continuous in a_{-j},

(iv) $B_j^P(a_{-j})$ is lower hemicontinuous in a_{-j} over $a_{-j} \in \widetilde{A}_{-j}$,[23]

(v) $\pi_j(a_j, a_{-j})$ is concave in a_j for every $a_{-j} \in A_{-j}$,[24]

where $B_j^P(a_{-j}) \coloneqq \{a_j \in A_j : u_i(a_j, a_{-j}, s_{i*}) \geq d_{ij}(a_{-j}, s) \ \forall i \in N_j(\beta_{*j})\}$ and $\widetilde{A} \coloneqq \{a \in A : a_k \in B_k^P(a_{-j}) \ \forall k \in M\}$. Then, a pure Nash equilibrium in Nash bargains exists.

4 A comparison of NB and WAPP

This section characterizes NB mechanisms and compares them with WAPP.

[23] Lemma 1 in the Appendix provides sufficient conditions for condition (iv) to hold.

[24] Assumption (v) guarantees that the Nash product is quasi-concave in a_j.

4.1 A characterization of NB mechanisms

For every firm $j \in M$, $s \in S$ and $\alpha_{-j} \in \times_{k \neq j}\Delta(A_{-k})$ define $\mathcal{U}_j(\alpha_{-j}, s) := \{x \in \mathbb{R}^n : \exists \alpha_j \in \Delta(A_j)$ such that $u(\alpha_j, \alpha_{-j}, s) = x\}$, the portfolio profit possibility set given the actions of the other firms. $\mathcal{U}_j(\alpha_{-j}, s)$ is convex since $\Delta(A_j)$ is. I strengthen this by assuming that $\mathcal{U}_j(\alpha_{-j}, s)$ is strictly convex for every $s \in S$ and $\alpha_{-j} \in \times_{k \neq j}\Delta(A_k)$. Strict convexity guarantees that firm j's corporate control mechanism is regular in the following sense.[25]

Definition 11. Firm j's corporate control mechanism g_j is regular if for every $s \in S$ and $\alpha_{-j} \in \times_{k \neq j}\Delta(A_{-k})$ and every $\alpha_j, \alpha'_j \in g_j(\alpha_{-j}, s)$, $u(\alpha_j, \alpha_{-j}, s) = u\left(\alpha'_j, \alpha_{-j}, s\right)$.

Regularity requires that a firm's best response is unique up to payoff-equivalent actions. This can be seen as an inherent consistency property of the corporate control mechanism. It implies that firm's shareholders are not willing to agree to two different policies when one of the two policies is strictly preferred to the other by at least one shareholder.

I now define efficient mechanisms.

Definition 12. The corporate control mechanism g_j of firm j is efficient if there exists function $\widetilde{N} : \Delta^n \to N$ such that for every $s \in S$,

(i) *(a nonempty set of investors control the firm)* $\widetilde{N}(s_{*j}) \neq \emptyset$,

(ii) *(only the firm's shareholders may control the firm)* for every $i \in N$, $s_{ij} = 0 \implies i \notin \widetilde{N}(s_{*j})$,

(iii) *(the firm is efficiently controlled)* for every $\alpha_{-j} \in \times_{k \neq j}\Delta(A_{-k})$, there do not exist $\alpha_j \in g_j(\alpha_{-j}, s)$ and $\alpha'_j \in \Delta(A_j)$ such that $u_i(\alpha'_j, \alpha_{-j}, s_{i*}) \geq u_i(\alpha_j, \alpha_{-j}, s_{i*})$ for all $i \in \widetilde{N}(s_{*j})$ with at least one strict inequality,

(iv) *(controlling shareholders only care about firm j's action so long as it affects their portfolio profits)* there do not exist $\alpha_j \in g_j(\alpha_{-j}, s)$ and $\alpha'_j \in \Delta(A_j) \setminus g_j(\alpha_{-j}, s)$ such that $u_i(\alpha_j, \alpha_{-j}, s_{i*}) = u_i\left(\alpha'_j, \alpha_{-j}, s_{i*}\right)$ for all $i \in \widetilde{N}(s_{*j})$.

A corporate control mechanism is efficient if under any ownership structure there is a subset $\widetilde{N}(s_{*j})$ of the shareholders of firm j that collectively and efficiently control the firm

[25] For $s \in S$ and $\alpha_{-j} \in \times_{k \neq j}\Delta(A_{-k})$ such that there exists $u \in \mathcal{U}_j(\alpha_{-j}, s)$ satisfying $u \gg d_{*j}$, NB is regular without strict convexity of $\mathcal{U}_j(\alpha_{-j}, s)$ being necessary, provided that the Nash product is strictly quasiconcave in u.

j. These controlling shareholders never choose actions for firm j that are (weakly) Pareto dominated in the sense that another action could do at least as well for all the controlling shareholders and strictly better for at least one of them. Also, if they are willing to select a certain (efficient) action α_j for firm j, then they are also willing to choose any other action that delivers the same payoff to each of the controlling shareholders as α_j does. Proposition 2 then characterizes NB mechanisms.

Proposition 2. Let firm j's corporate control mechanism be g_j.

(i) If g_j is WAPP, then it is regular and efficient.

(ii) g_j is regular and efficient if and only if it is NB.

Limiting attention to NB mechanisms amounts to considering all efficient and regular mechanisms, which is a superset of WAPP mechanisms.[26] Of course, efficiency is a minimal condition and there are additional desirable properties. Subsection 4.3 will argue that using NBRD is a way to proceed that overcomes the shortcomings of WAPP mechanisms.

4.2 First order conditions and the endogeneity of control weights

For notational simplicity, assume corporate control mechanisms prescribe a single (pure) action. If A_j is a convex subset of a Euclidean space with $\text{WAPP}_{\gamma_{*j}}$ pinned down by the first order condition (FOC), then

$$\sum_{i\in N} \gamma_{ij}(s_{*j})\ \frac{\partial u_i(a_j,a_{-j},s_{i*})}{\partial a_j}\bigg|_{a_j=\text{WAPP}_{\gamma_{*j}}(a_{-j},s)} = \mathbf{0},$$

where $\partial u_i(a_j,a_{-j},s_{i*})/\partial a_j$ denotes the gradient with respect to a_j, or equivalently

$$\frac{\partial \pi_j(a_j,a_{-j})}{\partial a_j}\bigg|_{a_j=\text{WAPP}_{\gamma_{*j}}(a_{-j},s)} + \sum_{k\in M\setminus\{j\}} \lambda_{jk}(s)\ \frac{\partial \pi_k(a_j,a_{-j})}{\partial a_j}\bigg|_{a_j=\text{WAPP}_{\gamma_{*j}}(a_{-j},s)} = \mathbf{0}.$$

An analogous FOC holds for $\text{NB}_{\beta_{*j},d_{*j}}$. When $u_i(\text{NB}_{\beta_{*j},d_{*j}}(a_{-j},s),a_{-j},s_{i*}) > d_{ij}(a_{-j},s)$ for every $i \in N_j(\beta_{*j})$, define the (normalized, unit-free) disagreement-adjusted bargaining

[26] Regularity is needed because there are efficient—but not regular—mechanisms that prescribe multiple actions (*i.e.*, $|g_j(\alpha_{-j},s)| > 1$ for some (α_{-j},s)) which are not all maximizers of the same Nash product. In other words, a Nash product picks a single point on the Pareto frontier, while an efficient—but not regular—mechanism can pick any subset of the frontier.

power of investor i over firm j to be given by

$$\widetilde{\gamma}_{ij}(\alpha_{-j},s) := \frac{\frac{\beta_{ij}(s_{*j})}{u_i\left(\text{NB}_{\beta_{*j},d_{*j}}(\alpha_{-j},s),\alpha_{-j},s_{i*}\right)-d_{ij}(\alpha_{-j},s)}}{\sum_{k\in N_j(\beta_{*j})}\frac{\beta_{kj}(s)}{u_k\left(\text{NB}_{\beta_{*j},d_{*j}}(\alpha_{-j},s),\alpha_{-j},s_{k*}\right)-d_{kj}(\alpha_{-j},s)}}.$$

$\widetilde{\gamma}_{ij}(\alpha_{-j},s)$ accounts for the concavity of the Nash product in $u_i\,(\alpha_j,\alpha_{-j},s_{i*})$. It measures shareholder control accounting for the fact that the further $u_i\,(\alpha_j,\alpha_{-j},s_{i*})$ is from $d_{ij}(\alpha_{-j},s)$, the more investor i has to lose in case of disagreement, which compromises her bargaining power. The FOC then reads

$$\sum_{i\in N_j(\beta_{*j})}\widetilde{\gamma}_{ij}(a_{-j},s)\left.\frac{\partial u_i(a_j,a_{-j},s_{i*})}{\partial a_j}\right|_{a_j=\text{NB}_{\beta_{*j},d_{*j}}(a_{-j},s)}=\mathbf{0},$$

or equivalently

$$\left.\frac{\partial\pi_j(a_j,a_{-j})}{\partial a_j}\right|_{a_j=\text{NB}_{\beta_{*j},d_{*j}}(a_{-j},s)}+\sum_{k\in M\setminus\{j\}}\widetilde{\lambda}_{jk}(\alpha_{-j},s)\left.\frac{\partial\pi_k(a_j,a_{-j})}{\partial a_j}\right|_{a_j=\text{NB}_{\beta_{*j},d_{*j}}(a_{-j},s)}=\mathbf{0},$$

where $\widetilde{\lambda}_{jk}(a_{-j},s) := \sum_{i\in N_j(\beta_{*j})}\widetilde{\gamma}_{ij}(a_{-j},s)s_{ik}/\sum_{i\in N_j(\beta_{*j})}\widetilde{\gamma}_{ij}(a_{-j},s)s_{ij}$ is the weight firm j *locally* places on firm k's profit.

A few comments are in place. In the WAPP mechanism, the control weights, γ_{*j}, do not depend on π or α_{-j}.[27] Indeed, a natural and simple way in which γ_{*j} can depend on either seems hard to find within the context of WAPP. On the other hand, NB provides us with a richer language than WAPP. The control power of each shareholder can depend (through d_{*j}) on both market conditions (*i.e.*, π), such as demand market demand and technology, and the other firms' actions.

Nevertheless, one should be cautious in interpreting $\widetilde{\gamma}$ as control weights. Although both γ_{ij} and $\widetilde{\gamma}_{ij}$ capture how strongly firm j's behavior will adapt to accommodate changes in the preferences of shareholder i (e.g., due to a stock trade performed by shareholder i), $\widetilde{\gamma}_{ij}$ is only valid for *local* changes in preferences. Thus, while γ_{ij} could be seen as a measure how strongly shareholder i's interests are represented in firm j's action, this is not true of $\widetilde{\gamma}_{ij}$. For example, if i exercises a lot of control over firm j (e.g., β_{ij} high), this can translate into a disproportionately high $u_i - d_{ij}$ (compared to that of shareholders

[27] To the best of my knowledge, the possibility of such dependence has not been considered in the literature.

with less control over firm j), which tends to decrease $\widetilde{\gamma}_{ij}$.[28]

Last, even though NB is more general than WAPP, the two obtain a similar characterization of an interior equilibrium. Thus, comparative statics under NB will also be valid under WAPP with the only difference that the corresponding WAPP control weights should be used. Of course, this comes with a loss in tractability due to the endogeneity of control weights under NB.

4.3 WAPP versus NBRD: ownership dispersion, theoretical predictions and policy implications

This section compares WAPP and NBRD. First, it shows that NBRD is more flexible than WAPP in accounting for ownership dispersion. Second, it shows that the two models can give rise to significantly different theoretical predictions and policy implications.

4.3.1 Ownership dispersion and modeling proportional control

I now define ownership structure rearrangements, which will be used in a behavioral definition of proportional control.

Definition 13. For every $v \in \left\{w \in \mathbb{R}^m_+ : w_j = 1\right\}$ and $s \in S$ define

$$p_j(v,s) \coloneqq \sum_{i \in \left\{i' \in N_j(s_{*j}) : \lambda_{i';j*} = v\right\}} s_{ij},$$

the total amount of shares held in firm j by shareholders that place weights v to the firms' profits (with the weight to firm j's profit normalized to 1). For any $s,s' \in S$ we say that s' is a rearrangement of s for firm j if for every $v \in \left\{w \in \mathbb{R}^m_+ : w_j = 1\right\}$, $p_j(v,s') = p_j(v,s)$.

$p_j(v,s)$ is the proportion of firm j shareholders that assign weights v to the firms' profits. The following definition describes proportional control using $p_j(v,s)$.

Definition 14. Firm j's corporate control mechanism g_j exhibits proportional control if for every $\alpha_{-j} \in \times_{k \neq j} \Delta(A_k)$ and for every $s,s' \in S$ such that s' is a rearrangement of s for firm j, $g_j\left(\alpha_{-j},s'\right) = g_j\left(\alpha_{-j},s\right)$.

[28] A similar comment applies to $\widetilde{\lambda}_{jk}(\alpha_{-j},s)$ and $\lambda_{jk}(s)$. Although both $\widetilde{\lambda}_{jk}(\alpha_{-j},s)$ and $\lambda_{jk}(s)$ capture how much firm j cares about the effect it has on firm k's profit, $\widetilde{\lambda}_{jk}(\alpha_{-j},s)$ is only valid locally.

This definition says that control of a firm is proportional when for any possible vector v of weights that the firm may assign to all firms' profits, only the proportion $p_j(v,s)$ of shareholders corresponding to weight vector v matters for firm conduct. Thus, for example "splitting" a shareholder into two shareholders—so that each of the two shareholders has half of the shares that the initial shareholder had in each firm—does not change firm conduct.

Proposition 3. Firm j's corporate control mechanism g_j exhibits proportional control if it is (i) NBRD with proportional bargaining power and lottery weight functions $\beta_{*j} = \beta_{*j}^P$ and $\delta_{*j} = \delta_{*j}^P$, or (ii) WAPP with $\gamma_{*j} = \gamma_{*j}^{sp-0}$ (*i.e.*, firm j maximizes the unweighted average of its shareholders' portfolio profits).

Under NB, the proportional control assumptions on β_{*j} and δ_{*j} indeed correspond to the behavioral definition of proportional control. On the other hand, this is not the case with WAPP for $\gamma_{*j}^{sp-\theta}$ (including $\theta = 1$, which corresponds to proportional γ_{*j}) unless $\theta = 0$. The WAPP mechanism by construction places an excessively high weight on the importance of concentration or dispersion of shares across shareholders within a firm.

Brito et al. (2023) argue that the WAPW formulation copes better with ownership dispersion than WAPP. Indeed, this is the case with their WAPW formulation that leads to γ^{sp-0}, which however gives rise to an arguably even more important issue. It specifies $\gamma_{ij}(s_{*j}) = |N_j(s_{*j})|^{-1}$ for every shareholder i of firm j, so that the firm maximizes the *unweighted* average of its shareholders' portfolio profits. This can be particularly problematic, as under proportional $\widehat{\gamma}_{*j}$ (which corresponds to γ_{*j}^{sp-0}) firm j assigns weight

$$\sum_{i \in N_j(\widehat{\gamma}_{*j})} \widehat{\gamma}_{ij}(s_{*j})\lambda_{i;jk} = \sum_{i \in N_j(\widehat{\gamma}_{*j})} s_{ij}\frac{s_{ik}}{s_{ij}} = \sum_{i \in N_j(\widehat{\gamma}_{*j})} s_{ik}$$

to firm k's profit, which is high. It is equal to 1 simply if firm k's shareholders are a subset of firm j's shareholders (*i.e.*, $N_k(\widehat{\gamma}_{*k}) \subseteq N_j(\widehat{\gamma}_{*j})$). If every firm follows this mechanism and every investor has some (however small or large) amount of shares in every firm in the industry, then maximization of aggregate industry profits by the firms (with the firms effectively acting as a multi-plant monopolist) will be an equilibrium. To see why this is unrealistic, start from $s = I_n$, where I_n the identity matrix (*i.e.*, each firm is owned by a unique shareholder). If one then slightly perturbs s, so that each investor has some amount of shares in each firm, the implication that firms will collectively act as a monopolist is unrealistic.[29]

More generally, the WAPW formulation can suffer from firm behavior being excessively affected by the extreme preferences of some shareholders. As s_{ij} decreases, the control $\widehat{\gamma}_{ij}$ of shareholder i decreases but at the same time the weight $\lambda_{i;jk}$ that she wants firm j to assign to firm k's profits increases at an increasing rate with $\lim_{s_{ij}\downarrow 0} \lambda_{i;jk} = \infty$.[30]

A duopoly example Consider a duopoly with the following symmetric ownership, WAPP control power and NBRD bargaining power and lottery weight structures, (s,γ,β,δ). Each firm has one common owner and $(n-1)/2$ non-common owners, where n the total number of investors. I study comparative statics with respect to n and thus parametrize objects by n (e.g., write $\gamma(n)$ instead of $\gamma(s)$).

$$s(n) = \begin{bmatrix} \sigma & \sigma \\ \frac{2(1-\sigma)}{n-1} & 0 \\ \vdots & \vdots \\ \frac{2(1-\sigma)}{n-1} & 0 \\ 0 & \frac{2(1-\sigma)}{n-1} \\ \vdots & \vdots \\ 0 & \frac{2(1-\sigma)}{n-1} \end{bmatrix}, \gamma(n) = \beta(n) = \delta(n) = \begin{bmatrix} \gamma_{11}(n) & \gamma_{11}(n) \\ \frac{2(1-\gamma_{11}(n))}{n-1} & 0 \\ \vdots & \vdots \\ \frac{2(1-\gamma_{11}(n))}{n-1} & 0 \\ 0 & \frac{2(1-\gamma_{11}(n))}{n-1} \\ \vdots & \vdots \\ 0 & \frac{2(1-\gamma_{11}(n))}{n-1} \end{bmatrix}.$$

Since $\gamma(n) = \beta(n) = \delta(n)$ I will use the γ notation also in NBRD. Claim 1 then studies how firm behavior changes under WAPP as n (and thus the dispersion of non-common owners) increases.

Claim 1. Consider the duopoly as defined above with $\sigma, \gamma_{11}(n) \in (0,1)$ for $n \in \{3,5,7,\dots\}$.[31] Let the firms' corporate control mechanisms be $g_1 = \text{WAPP}_{\gamma_{*1}(n)}$ and $g_2 = \text{WAPP}_{\gamma_{*2}(n)}$.

(i) If we ignore the integer constraint on n and differentiate $\gamma_{11}(n)$ with respect to n, then $\lambda_{12}(n) = \lambda_{21}(n)$ is increasing (resp. decreasing) in n if and only if

$$\frac{\partial \gamma_{11}(n)}{\partial n} \frac{n}{\gamma_{11}(n)} + \frac{2n}{n-1}(1-\gamma_{11}(n)) \overset{\text{(resp. <)}}{>} 0.$$

[29] A similar perturbation can be made if we start from each firm being held by multiple (rather than a single) non-common owners.

[30] As we will see next, the γ^{mB} formulation does not satisfactorily handle ownership dispersion either, and at the same time has other unappealing properties.

[31] If $\gamma_{11}(n) = 1$ for some n, then trivially $\lambda_{12}(n) = \lambda_{21}(n) = 1$ for that n.

(ii) If $\lim_{n\to\infty} n\gamma_{11}(n) = \infty$, then $\lim_{n\to\infty} \lambda_{12}(n) = \lim_{n\to\infty} \lambda_{21}(n) = 1$.

(iii) If the control weights are sp–θ, then $\lambda_{12}(n) = \lambda_{21}(n)$ goes to 1 (resp. σ) as $n \to \infty$ if $\theta > 0$ (resp. $\theta = 0$).

(iv) (Dubey and Shapley, 1979) Under Banzhaf control weights $\lim_{n\to\infty} \gamma_{11}^B(n) = 1$. Thus, $\lim_{n\to\infty} \lambda_{12}(n) = \lim_{n\to\infty} \lambda_{21}(n) = 1$.

(v) If the control weights are modified Banzhaf, then $\lim_{n\to\infty} \lambda_{12}(n) = \lim_{n\to\infty} \lambda_{21}(n) = 1$.

(vi) When $\lambda_{12}(n) = \lambda_{21}(n) = 1$, each firm maximizes aggregate industry profits, so the two-plant monopoly solution is an equilibrium.

Part (i) shows that even if the common owner's share of control $\gamma_{11}(n)$ does not increase with n, firms may internalize each other's profits by more and more as n increases. Part (ii) examines the limiting case where non-common owners become very dispersed ($n \to \infty$). Unless $\gamma_{11}(n)$ goes to 0 at a rate of at least $1/n$, as $n \to \infty$ WAPP assigns all control of both firms to the common owner giving rise to a two-plant monopoly (part (vi)).[32] No reasonable assumption on γ's can overcome this issue as seen in parts (iii)-(v). This is because letting $\gamma_{11}(n)$ go to 0 (and fast) gives rise to other issues.[33]

Figure 1 plots the control power of investor 1, $\gamma_{11}(n)$, and the weight each firm assigns to the other firm's profit, $\lambda_{12}(n) = \lambda_{21}(n)$, under alternative specifications of WAPP. Apart from inadequately handling ownership dispersion, Banzhaf and modified Banzhaf also generate unintuitive non-monotonicities in $\gamma_{11}(n)$ and $\lambda_{12}(n)$. $\gamma^{sp-\theta}$ with $\theta < 1$ dampens the unnatural effects of ownership dispersion under WAPP without giving rise to non-monotonicities. In the limit as $n \to \infty$ only the unrealistic specification of $\theta = 0$ combats the effects of ownership dispersion. However, if control is not proportional, it is more natural to think of larger shareholders having more than proportional control (*i.e.*, $\theta > 1$). This emphasizes the tension between the behavioral definition of proportional control and the definition of proportional control in terms of parameters in the WAPP model.

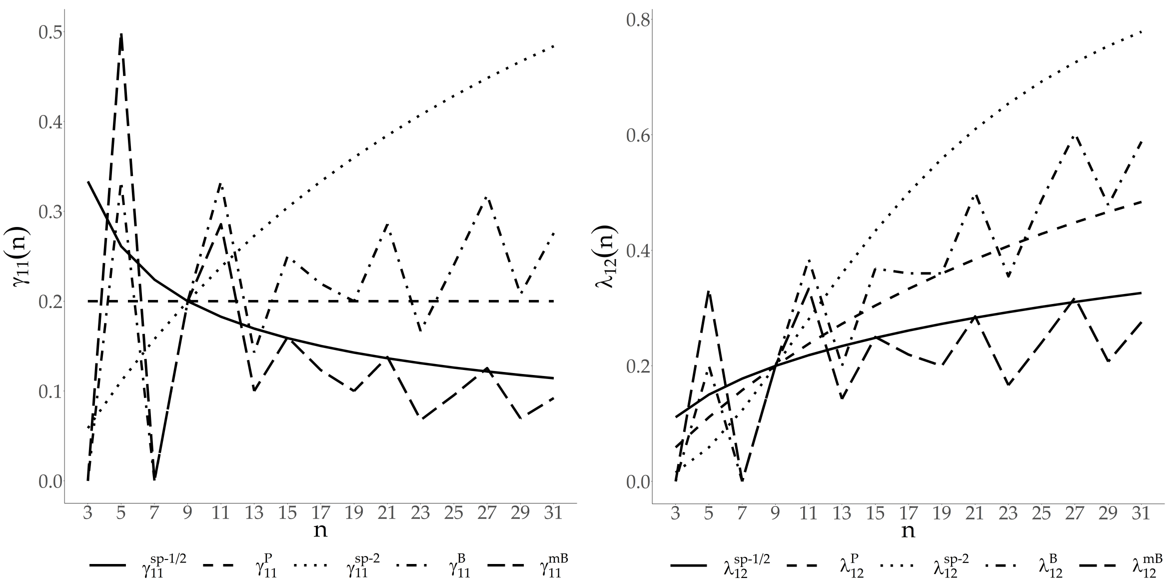
Claim 2 then studies how firm behavior changes under NBRD as n increases.

[32] The two firms do not jointly maximize profits but rather each of them chooses its action to maximize aggregate industry profits *given* the other firm's action. Thus, although two-plant monopoly indeed is an equilibrium, there may also be other equilibria (e.g., see Vives and Vravosinos, 2023).

[33] As we have seen, γ^{sp-0} is unrealistic.

Figure 1: Common owner control power and profit weights in a duopoly with WAPP and varying non-common ownership dispersion

(a) Control power of common owner **(b)** Profit weight on other firm



Claim 2. Consider the duopoly as defined above with $\sigma \in (0{,}1)$ and $n \in \{3{,}5{,}7,\ldots\}$. Let the firms' corporate control mechanisms be $g_1 = \text{NB}_{\beta_{*1}(n),d_{*1}(n)}$ and $g_2 = \text{NB}_{\beta_{*2}(n),d_{*2}(n)}$ with $d_{*1}(n)$ and $d_{*2}(n)$ RD with lottery weights $\delta_{*1}(n)$ and $\delta_{*2}(n)$, respectively.

(i) If $\lim_{n\to\infty}\gamma_{11}(n) = 1$ (resp. 0), then as $n \to \infty$, in the limit each firm's objective is to maximize aggregate industry profits (resp. maximize own profit).

(ii) Under sp–θ $\gamma(n)$, for

(a) $\theta \in [0,1)$, $\lim_{n\to\infty}\gamma_{11}^{sp-\theta}(n) = 0$,

(b) $\theta = 1$, $\gamma_{11}^{sp-1}(n) = \sigma$ for every n so each firm's behavior is invariant to n,

(c) $\theta > 1$, $\lim_{n\to\infty}\gamma_{11}^{sp-\theta}(n) = 1$.

There is an intuitive connection between parameter values and behavioral properties of the resulting NBRD mechanism. Introducing convexity through $\theta > 1$, which gives large shareholders more than proportional control as measured by γ_{11}, indeed translates to the large shareholder gaining complete control of each firm as remaining ownership is dispersed among infinitely many shareholders. The converse is true under $\theta < 1$. Thus, as $n \to \infty$, the two firms behave as a two-plant monopolist (resp. compete with each of them maximizing own profit) if $\theta > 1$ (resp. $\theta < 1$). When $\theta = 1$, the control weights (*i.e.*, β_{ij} and δ_{ij}) of each shareholder are proportional to the number of shares she holds, and as shown in Proposition 3, this exactly corresponds to the behavioral definition of proportional control.

4.3.2 Competitive effects of common ownership and policy implications

Finally, I compare WAPP and NBRD in terms of theoretical predictions and policy implications. Specifically, I look at how market outcomes change as an investor varies the degree of diversification of a fixed number of shares across the industry.

Consider a homogeneous product Cournot duopoly ($m = 2$) with 3 investors ($n = 3$), linear inverse demand $P(Q) = \max\{10 - Q,0\}$ and symmetric linear cost functions $C_1(q_1) = q_1$, $C_2(q_2) = q_2$. Under both NBRD and WAPP, let control be proportional $\beta(s) = \gamma(s) = \delta(s) = s$, and the ownership structure be

$$s = \begin{bmatrix} s_{11} & 0.45 - s_{11} \\ 1 - s_{11} & 0 \\ 0 & 0.55 + s_{11} \end{bmatrix},$$

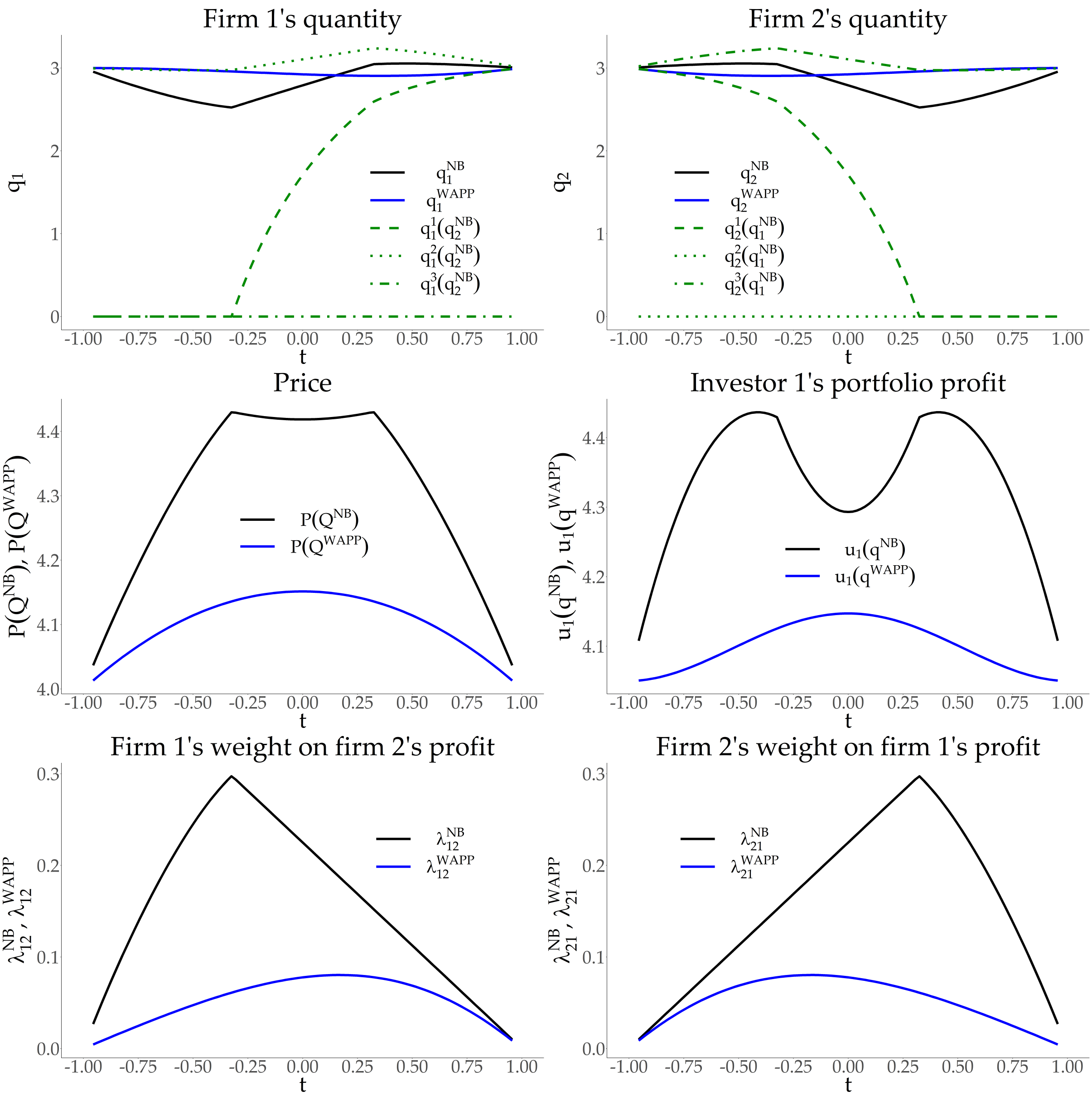
which is indexed by the shares s_{11} of investor 1 in firm 1.

The two firms are equally efficient and investor 1 (e.g., a large fund) can choose how to distribute her total holdings of 0.45 in the industry between the two firms. Investors 2 and 3 are passive in that they are indifferent towards the capital they invest in the firms. The fund can buy shares of either firm at the same price and the rest of the capital is provided by investors 2 and 3. Define the normalized value $t := (s_{11} - 0.225)/0.225 \in [-1,1]$ measuring what firm the fund's holdings are concentrated in. The closer t is to 0, the higher is the fund's diversification; for $t = 0$ the equilibrium is symmetric. As t increases investor 1's holdings become more concentrated in firm 1.

Think of a policy that limits the degree of common ownership an investor can have within the industry; it specifies some $\tau \in [0,1]$ and requires that $t \in [-1, -\tau] \cup [\tau,1]$. Figure 2 shows equilibrium results under NBRD and WAPP.

If the fund only cares to maximize its portfolio profit, then under WAPP it will choose t as close to 0 as possible. Thus, the price is decreasing in the restrictiveness τ of the policy. However, under NBRD the fund picks t as close as possible to either of the two peaks (in its portfolio profit) as possible, so that the price is first constant and then decreasing in τ. Therefore, a policy that is effective in increasing consumer welfare under

Figure 2: Equilibrium with a large fund and two undiversified passive investors for varying levels of diversification by the fund



Note: black lines represent equilibrium values under NBRD; blue ones under WAPP. Green lines show the most preferred quantity of each shareholder for each firm with the competitor's quantity taken as given (fixed at its equilibrium value). The bottom two panels plot λ_{12},λ_{21} (under WAPP) and $\widetilde{\lambda}_{12}$,$\widetilde{\lambda}_{21}$ (under NBRD).

WAPP may be ineffective under NBRD.[34]

Consider now an alternate scenario where the fund only cares to maximize its portfolio diversification, that is $\min |t|$, in order for example to mitigate risk or track an industry index. Then, under WAPP, the price is decreasing in τ. However, under NBRD, the price is first increasing and then decreasing in τ. Thus, a policy that is effective under WAPP may in fact harm consumer welfare under NBRD.

The differences in predictions between WAPP and NBRD are due to the differences (between the two mechanisms) in magnitudes of the various channels through which a change in t affects equilibrium outcomes. As t changes, both the fund's preferences and the division of power within each firm change.

Under WAPP, as t (*i.e.*, s_{11}) increases, the degree to which the fund wants firm 1 (resp. 2) to internalize firm 2's (resp. 1's) profits decreases (resp. increases), which tends to shift production towards firm 1. On the other hand, as t increases investor 2's control of firm 1 decreases, and investor 3's control of firm 2 increases, which tend to shift production towards firm 2. Under WAPP, around $t = 0$, the latter effects dominate, so that firm 2's quantity increases with t, while the quantity of firm 1 decreases making it unprofitable for the fund to pick $t \neq 0$. Also, firm 1's quantity increases faster than firm 2's quantity decreases with t (around $t = 0$), and the price has a global maximum at $t = 0$ under.

However, under NBRD, as t increases (around $t = 0$), production shifts towards firm 1, which is in the interest of the fund when $t > 0$. This makes it profitable for the fund to pick $t \neq 0$. Also, firm 1's quantity increases more slowly than firm 2's quantity decreases with t (around $t = 0$), so that the price has a local minimum at $t = 0$ under NBRD.

Similarly, based on WAPP a consumer-welfare-maximizing regulator would want to block a trade that brings t from -0.25 to 0, even though this trade would increase consumer welfare under NBRD.

Last, notice that the graphs of control weights γ and $\widetilde{\gamma}$ differ between WAPP and NBRD. These weights capture the extent to which changes in investor preferences (e.g., due to a stock trade) will be accommodated by each firm. Thus, the WAPP and NBRD models will give different predictions regarding stock trade effects.

[34] Remember that consumer surplus is increasing in the total quantity (and thus decreasing in the price) in a homogeneous product market.

5 Application: homogeneous product Cournot oligopoly

This section characterizes the Nash-in-Nash equilibrium of a homogeneous product Cournot oligopoly and studies how changes in corporate control affect equilibrium outcomes.[35]

5.1 A Nash-in-Nash model of Cournot oligopoly with common ownership

There is a set N of n firms producing a homogeneous good. Each firm j chooses its production quantity q_j simultaneously with the other firms. Denote by $w_j \equiv q_j/Q$ firm j's market share of the total quantity $Q \coloneqq \sum_{k=1}^{n} q_k$. q_{-j} denotes the production profile of the firms other than j, and $Q_{-j} \coloneqq \sum_{k\neq j}^{n} q_k$. Firm j's production cost is given by the twice-differentiable function $C_j : \mathbb{R}_+ \to \mathbb{R}_+$ with $C_j'(q_j) > 0$ globally.

The twice-differentiable inverse demand function $P(Q)$ satisfies $P'(Q) < 0 \; \forall Q \in \left[0,\overline{Q}\right)$, where $\overline{Q} \in (0,+\infty]$ is such that $P(Q) > 0 \iff Q \in \left[0,\overline{Q}\right)$. $\eta(Q) \coloneqq -P/(QP')$ denotes the elasticity of demand. Firm j's profit is given by $\pi_j(q) \coloneqq q_j P(Q) - C_j(q_j)$.

Define the following index of the weight firm j places on other firms' profits

$$\overline{\lambda}_j(q,s) \coloneqq \sum_{k\in M\setminus\{j\}} w_k \widetilde{\lambda}_{jk}(q_{-j},s) \equiv \sum_{k\in M\setminus\{j\}} w_k \frac{\sum_{i\in N_j(\beta_{*j})} \widetilde{\gamma}_{ij}(q_{-j},s) s_{ik}}{\sum_{i\in N_j(\beta_{*j})} \widetilde{\gamma}_{ij}(q_{-j},s) s_{ij}}.$$

Similarly, for each pair of distinct firm's j and k and each shareholder i of firm j define $\overline{\lambda}_{i;j}(q,s_{i*}) \coloneqq \sum_{k\in M\setminus\{j\}} w_k \lambda_{i;jk}$, an index of the weight shareholder i wants firm j to place on other firms' profits.

Define also the bargaining-adjusted (i) Herfindahl-Hirschman Index (HHI) of market shares, (ii) MHHIΔ, and (iii) modified HHI, (iv) weighted average Lerner index LI, respectively given by

$$\text{HHI}(q) \coloneqq \sum_{j\in M} w_j^2, \quad \text{MHHI}\Delta(q,s) \coloneqq \sum_{j\in M} w_j \overline{\lambda}_j(q,s),$$
$$\text{MHHI}(q,s) \coloneqq \text{HHI}(q) + \text{MHHI}\Delta(q,s), \quad \overline{\text{LI}}(q) \coloneqq \sum_{j=1}^{m} w_j \frac{P(Q) - C_j'(q_j)}{P(Q)}.$$

[35] As seen in section 4, the analysis is also valid under WAPP.

5.2 Nash-in-Nash equilibrium characterization

Let $\widetilde{S} \subseteq S$ be an open subset of S such that for every $s \in \widetilde{S}$, there is a unique and interior equilibrium q^* where $u\left(\mathrm{NB}_{\beta_{*j},d_{*j}}\left(q^*_{-j},s\right),q^*_{-j},s\right) \gg d_{*j}(q^*_{-j},s)$ for every firm $j \in M$. $q^* : \widetilde{S} \to \mathbb{R}^m_{++}$ returns this equilibrium as a function of s.[36] Similarly, write $Q^* \equiv \sum_{k\in M} q^*_k$, $w^*_j := q^*_j/Q^*$. To simplify notation, define also $\gamma^*_{ij}(s) := \widetilde{\gamma}_{ij}(q^*_{-j}(s),s)$, $\lambda^*_{jk}(s) := \widetilde{\lambda}_{jk}(q^*_{-j}(s),s)$, $\overline{\lambda}^*_j(s) := \overline{\lambda}_j(q^*(s),s)$, $\overline{\lambda}^*_{i;j}(s) := \overline{\lambda}_{i;j}(q^*(s),s_{i*})$ for every investor $i \in N$ and pair of distinct firms $j,k \in M$. These functions give the equilibrium values of the corresponding objects as functions of the ownership structure. $q^*(s)$ is then pinned down by the following FOCs:

$$f(q,s) := \left(\sum_{i\in N_1(\beta_{*1})} \gamma^*_{i1}(s)\frac{\partial u_i(q,s_{i*})}{\partial q_1} \quad \dots \quad \sum_{i\in N_m(\beta_{*m})} \gamma^*_{im}(s)\frac{\partial u_i(q,s_{i*})}{\partial q_m}\right)\Bigg|_{q=q^*(s)} = \mathbf{0}.$$

Denote the Jacobian of $f(q,s)$ (with respect to q) by $J(q,s)$. An interior, regular equilibrium is then defined as follows.

Definition 15. An equilibrium q^* is called interior and regular if (i) $q^* \gg \mathbf{0}$, (ii) for every firm $j \in M$, $d_{N_j(\beta_{*j})j}(q^*_{-j},s) \ll u_{N_j(\beta_{*j})}\left(q^*_j,q^*_{-j},s\right)$, and (iii) $J(q^*,s)$ is negative definite.

It is a maintained assumption that the equilbirium is interior and regular. Proposition 4 derives the equilibrium markup of each firm and the relationship between the weighted average Lerner index and the MHHI.

Proposition 4. In equilibrium for every firm $j \in M$ it holds that

$$\frac{P(Q^*) - C'_j(q^*_j)}{P(Q^*)} = \frac{w^*_j + \overline{\lambda}^*_j(s)}{\eta(Q^*)}.$$

The weighted average Lerner Index is $\overline{\mathrm{LI}}(q^*) = \mathrm{MHHI}(q^*,s)/\eta(Q^*)$.

5.3 Competitive effects of changes in corporate control

Consider an exogenous change in an investor's control power over a firm.

Definition 16. An exogenous increase (resp. decrease) in investor i's control over firm j at $s \in S \times \mathbb{R}^m_+$ is a change in the corporate control mechanism of firm j so that $\beta_{ij}(s_{*j})$

[36] I will sometimes simply write q^* instead of $q^*(s)$.

changes infinitesimally by $d\beta_{ij} >$ (resp. $<$) 0 with all else kept constant.[37]

Proposition 5 then studies the effects of a change in a shareholder's control over a firm.

Proposition 5. An exogenous increase (resp. decrease) in investor i's control over firm j causes firm j's quantity to change in the direction (resp. direction opposite to the one) preferred by shareholder i, that is

$$\operatorname{sgn}\left\{\frac{dq_j^*}{d\beta_{ij}}\right\} = \operatorname{sgn}\left\{\left.\frac{\partial u_i\left(q, s_{i*}\right)}{\partial q_j}\right|_{q=q^*}\right\} = \operatorname{sgn}\left\{\overline{\lambda}_j^*(s) - \overline{\lambda}_{i;j}^*(s)\right\}.$$

Proposition 5 shows that if a firm is underproducing (resp. overproducing) relative to a shareholder's preferences and that shareholder's control over that firm increases, then the firms quantity will increase (resp. decrease). The proposition also provides an intuitive measure of whether the firm is under- or overproducing relative to the investor's preferences. It underproduces (resp. overproduces) if its (local) weighted average Edgeworth coefficient $\overline{\lambda}_j^*(s)$ is higher (resp. lower) than the shareholder's weighted Edgeworth coefficient.

A policy proposal by Posner et al. (2017) is to require institutional investors to be passive if they accumulate large amounts of stock in multiple competing firms. Such a policy can be understood as setting $\beta_{ij} = 0$ for an investment fund i and every firm j. Provided that total quantity changes in the same direction as firm j's quantity, this policy will indeed increase consumer welfare if $\overline{\lambda}_{i;j}^*(s) > \overline{\lambda}_j^*(s)$ along a path where β_{ij}'s go to 0 for every firm j.[38]

6 Conclusion

Both theoretical and empirical work has so far followed the weighted average portfolio profit (WAPP) model of O'Brien and Salop (2000) to model corporate control under common ownership. This paper has proposed an alternative model of corporate control modeling firm policy as the result of asymmetric Nash bargaining (NB) among shareholders.

[37]For the entries of β_{*j} to still sum up to 1, the other entries clearly need to decrease. However, this is just a normalization that does not affect the analysis, so it is ignored. Also, notice that an exogenous increase (resp. decrease) in d_{ij} will have the same qualitative effect as an increase (resp. decrease) in β_{ij}.

[38]Under WAPP, the total quantity changes in the same direction as firm j's quantity if the game is aggregative and the slope of each firm's best response function is higher than -1 (e.g., see Farrell and Shapiro, 1990; Vives, 1999). The game is aggregative if s is such that for every firm $j \in M$, $\lambda_{jk}(s) = \lambda_{jh}(s)$ for every pair of firms k,$h \in M \setminus \{j\}$.

It thereby extends the use of the Nash-in-Nash approach to the case of oligopolistic competition when within each firm shareholders bargain over firm conduct.

I have shown that NB is a rich framework within which a satisfying model of corporate control can be searched for. WAPP is indeed a special case of the NB model, yet with important deficiencies. I have argued that NB with random dictatorship disagreement payoffs (NBRD) overcomes the deficiencies of WAPP and gives rise to a natural connection between model parameters and properties of a firm's best response function. At the same time, adopting NBRD instead of WAPP leads to significantly different theoretical results and policy implications. Last, I have applied the Nash-in-Nash approach in a Cournot market to study the competitive effects of changes in corporate control offering a rationale for policy recommendations that would require institutional investors to be passive.

Future work can examine whether these differences between NBRD and WAPP extend beyond theoretical results to (structural) empirical estimates. Empirical studies can leverage the Nash-in-Nash approach to estimate the effects of common ownership and test the robustness of findings derived under the standard approach that uses WAPP (e.g., see Backus et al., 2021).

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Appendix A: proofs

Proof of Proposition 1 The game can be seen as a generalized game where the action constraint correspondence is $B_j^P(a_{-j},s) := \{a_j \in A_j : u(a_j,a_{-j},s) \geq d_{*j}(a_{-j},s)\}$. The proof is composed of three steps.

Step 1: $B_j^P(a_{-j},s)$ is

(i) non-empty by property (i) of disagreement payoffs of NB mechanisms,

(ii) compact as a closed subset of a compact set (since u is continuous in a_j),

(iii) upper hemicontinuous in a_{-j}, as a closed-valued correspondence to a compact space (e.g., see Corollary 9 in p.111, Aubin and Ekeland, 1984),

(iv) lower hemicontinuous in a_{-j} by assumption.

Also, the Nash product is continuous in a_j and a_{-j} given that u and d_{*j} are. It follows then by Berge's maximum theorem that $g_j(a_{-j},s)$ is an upper hemicontinuous, non-empty-valued and compact-valued correspondence.

Step 2: For any $i \in N$ and any $a_{-j} \in A_j$ we have that $u_i\left(\delta a_j + (1-\delta)a'_j,a_{-j},s_{i*}\right) - d_{ij}(a_{-j},s)$ is concave over $B_j^P(a_{-j},s)$. It follows that for any $i \in N_j(\beta_{*j})$ and any a_{-j}

$$\left(u_i\left(\delta a_j + (1-\delta)a'_j,a_{-j},s_{i*}\right) - d_{ij}(a_{-j},s)\right)^{\beta_{ij}(s_{*j})}$$

is concave (and thus log-concave) over $B_j^P(\alpha_{-j},s)$, since $a_j \mapsto u_i\left(\delta a_j + (1-\delta)a'_j,a_{-j},s_{i*}\right) - d_{ij}(a_{-j},s)$ is concave and $x \mapsto x_{ij}^{\beta}(s_{*j})$ is concave and increasing. Thus,

$$\prod_{i \in N_j(s)} \left(u_i\left(a_j,a_{-j},s_{i*}\right) - d_{ij}(a_{-j},s)\right)^{\beta_{ij}(s_{*j})}$$

is log-concave over $B_j^P(a_{-j},s)$ as a product of log-concave functions (and thus also quasi-concave in a_j for every a_{-j}). The product is also continuous in a_j and a_{-j}, and given also that $B_j^P(a_{-j},s)$ is convex for any $a_{-j} \in A_{-j}$, it follows that $g_j(a_{-j},s)$ is convex-valued.

Step 3: $G(a) := \times_{j \in M} g_j(a_{-j},s)$ is an upper hemicontinuous, non-empty-, compact- and convex-valued correspondence since g_j is for each $j \in M$. By Kakutani's fixed point theorem, G admits a fixed point, which is an equilibrium. **Q.E.D.**

Proof of Proposition 2 Part (i) Let g_j be WAPP with control power function γ_{*j}.

Regularity: Assume by contradiction that g_j is not regular. That is, there exist $s \in S$, $\alpha_{-j} \in \times_{k \neq j} \Delta(A_{-k})$ and $\alpha_j, \alpha'_j \in g_j(\alpha_{-j}, s)$, such that $u(\alpha_j, \alpha_{-j}, s) \neq u(\alpha'_j, \alpha_{-j}, s)$. Then, by strict convexity of $\mathcal{U}_j(\alpha_{-j}, s)$, for any $\lambda \in (0,1)$ it holds that

$$v := \lambda u(\alpha_j, \alpha_{-j}, s) + (1-\lambda) u(\alpha'_j, \alpha_{-j}, s) \in \text{int}(\mathcal{U}_j(\alpha_{-j}, s))$$

and thus there exists $v' \in \mathcal{U}_j(\alpha_{-j}, s)$ such that $v' \gg v$, or equivalently α^*_j such that $u(\alpha^*_j, \alpha_{-j}, s) \gg v$. But then $\sum_{i \in N} \gamma_{ij}(s_{*j}) u_i(\alpha^*_j, \alpha_{-j}, s_{i*})$ is higher than

$$\begin{aligned} &> \lambda \sum_{i \in N} \gamma_{ij}(s_{*j}) u_i(\alpha_j, \alpha_{-j}, s_{i*}) + (1-\lambda) \sum_{i \in N} \gamma_{ij}(s_{*j}) u_i(\alpha'_j, \alpha_{-j}, s_{i*}) \\ &= \sum_{i \in N} \gamma_{ij}(s_{*j}) u_i(\alpha_j, \alpha_{-j}, s_{i*}) = \sum_{i \in N} \gamma_{ij}(s_{*j}) u_i(\alpha'_j, \alpha_{-j}, s_{i*}), \end{aligned}$$

which contradicts that $\alpha_j, \alpha'_j \in g_j(\alpha_{-j}, s)$.

Efficiency: For every $s \in S$ define $\widetilde{N}(s_{*j}) = \{i \in N : \gamma_{ij}(s_{*j}) > 0\}$, and use $\widetilde{N}(s_{*j})$ to verify that g_j satisfies the efficiency conditions.

Part (ii) I know prove each direction of part (ii) separately.

($\Rightarrow$) Let g_j be regular and efficient, so that there exists function $\widetilde{N}(s_{*j})$ satisfying the the efficiency conditions. For every $(\alpha_{-j}, s) \in \times_{k \neq j} \Delta(A_{-k}) \times S$ let the bargaining power function be

$$\beta_{*j}(s_{*j}) := \frac{1}{\left|\widetilde{N}(s_{*j})\right|} \left(\mathbb{I}\left(1 \in \widetilde{N}(s_{*j})\right) \quad \ldots \quad \mathbb{I}\left(n \in \widetilde{N}(s_{*j})\right) \right),$$

where $\mathbb{I}$ the indicator function, and the disagreement payoff function be $d_{*j}(\alpha_{-j}, s) := u(\alpha_j(\alpha_{-j}, s), \alpha_{-j}, s)$ for some function $\alpha_j(\alpha_{-j}, s)$ that is a selection from $g_j(\alpha_{-j}, s)$, that is $\alpha_j(\alpha_{-j}, s) \in g_j(\alpha_{-j}, s)$. d_{*j} is well-defined since g_j is regular. Notice that by the way β_{*j} is defined, $N_j(\beta_{*j}) = \widetilde{N}(s_{*j})$. Then, it follows that

(a) any $\alpha_j \in g_j(\alpha_{-j}, s)$ achieves the maximum value of zero for the Nash product, so

$$g_j(\alpha_{-j}, s) \subseteq \underset{\alpha_j \in B_j(a_{-j}, s)}{\arg\max} \left\{ \prod_{i \in N_j(\beta_{*j})} \left(u_i(\alpha_j, \alpha_{-j}, s_{i*}) - d_{ij}(\alpha_{-j}, s)\right)^{\beta_{ij}(s_{*j})} \right\}.$$

(b) and by regularity of NB (shown below) any

$$\alpha_j,\alpha'_j \in \arg\max_{\alpha_j \in B_j(a_{-j},s)} \left\{ \prod_{i \in N_j(\beta_{*j})} \left(u_i\left(\alpha_j,\alpha_{-j},s_{i*}\right) - d_{ij}(\alpha_{-j},s)\right)^{\beta_{ij}(s_{*j})} \right\}.$$

satisfy $u_i\left(\alpha_j,\alpha_{-j},s_{i*}\right) = u_i\left(\alpha'_j,\alpha_{-j},s_{i*}\right)$ for every $i \in N_j(\beta_{*j})$. Therefore, by efficiency of g_j (condition (iv) in Definition 12)

$$g_j\left(\alpha_{-j},s\right) \supseteq \arg\max_{\alpha_j \in B_j(a_{-j},s)} \left\{ \prod_{i \in N_j(\beta_{*j})} \left(u_i\left(\alpha_j,\alpha_{-j},s_{i*}\right) - d_{ij}(\alpha_{-j},s)\right)^{\beta_{ij}(s_{*j})} \right\}.$$

($\Leftarrow$) Let g_j be NB with bargaining power function β_{*j}.

Regularity: I look at the following two cases separately.

Case 1: Consider the case where there exists $u \in \mathcal{U}_j\left(\alpha_{-j}, s\right)$ such that $u_i > d_{ij}$ for every $i \in N_j(\beta_{*j})$. The Nash product $\prod_{i \in N_j(\beta_{*j})}\left(u_i - d_{ij}\right)^{\beta_{ij}}$ is strictly quasiconcave in u where that inequality holds. Thus, since $\mathcal{U}_j\left(\alpha_{-j}, s\right)$ is convex for every $s \in S$ and $\alpha_{-j} \in \times_{k \neq j}\Delta(A_k)$, there exists at most one $u \in \mathcal{U}_j\left(\alpha_{-j}, s\right)$ that maximizes the Nash product.

Case 2: Consider the case where there does *not* exist $u \in \mathcal{U}_j\left(\alpha_{-j}, s\right)$ such that $u_i > d_{ij}$ for every $i \in N_j(\beta_{*j})$. Assume by contradiction that there are two distinct $u, u' \in \mathcal{U}_j\left(\alpha_{-j}, s\right)$ that maximize the Nash product (which achieves a value of zero). Then, by strict convexity of $\mathcal{U}_j\left(\alpha_{-j}, s\right)$, for any $\lambda \in (0,1)$ it holds that $v := \lambda u + (1-\lambda)u' \in \operatorname{int}\left(\mathcal{U}_j\left(\alpha_{-j}, s\right)\right)$, and thus there exists $v' \in \mathcal{U}_j\left(\alpha_{-j}, s\right)$ such that $v' \gg v$, or equivalently α^*_j such that $u\left(\alpha^*_j,\alpha_{-j}, s\right) \gg v$. But then, α^*_j makes the Nash product positive (thus higher than u and u' do), a contradiction.

Efficiency: For every $s \in S$ define $\widetilde{N}(s_{*j}) := \{i \in N : \beta_{ij}(s_{*j}) > 0\}$, and use $\widetilde{N}(s_{*j})$ to verify that g_j satisfies the efficiency conditions. **Q.E.D.**

Proof of Proposition 3 *Part (i)* Let g_j be NBRD with $\beta_{*j} = \beta^P_{*j}$ and $\delta_{*j} = \delta^P_{*j}$, that is, for every $s \in S$ and $\alpha_{-j} \in \times_{k \neq j}\Delta(A_{-k})$, $g_j\left(\alpha_{-j},s\right)$ is equal to

$$\arg\max_{\alpha_j \in B_j(a_{-j},s)} \left\{ \prod_{i \in N_j(s_{*j})} \left(u_i\left(\alpha_j,\alpha_{-j},s_{i*}\right) - \sum_{k \in N_j(s_{*j})} s_{kj} u_i\left(\alpha^d_j\left(\alpha_{-j},\frac{s_{k*}}{s_{kj}}\right),\alpha_{-j},s_{i*}\right)\right)^{s_{ij}} \right\}.$$

Now, notice that because of the linearity of $u_i(\alpha_j,\alpha_{-j},s_{i*})$ in s_{i*}, the Nash product above is equal to

$$\prod_{i\in N_j(s_{*j})} s_{ij}^{s_{ij}} \left(u_i\left(\alpha_j,\alpha_{-j},\frac{s_{i*}}{s_{ij}}\right) - \sum_{k\in N_j(s_{*j})} s_{kj} u_i\left(\alpha_j^d\left(\alpha_{-j},\frac{s_{k*}}{s_{kj}}\right),\alpha_{-j}\frac{s_{i*}}{s_{ij}}\right)\right)^{s_{ij}},$$

which (given that the positive multiplicative terms $s_{ij}^{s_{ij}}$ do not affect the extrema of the Nash product with respect to α_j) implies that $g_j(\alpha_{-j},s)$ is equal to

$$\operatorname*{arg\,max}_{\alpha_j\in B_j(a_{-j},s)} \left\{ \prod_{i\in N_j(s_{*j})} \left(u_i\left(\alpha_j,\alpha_{-j},\frac{s_{i*}}{s_{ij}}\right) - \sum_{k\in N_j(s_{*j})} s_{kj} u_i\left(\alpha_j^d\left(\alpha_{-j},\frac{s_{k*}}{s_{kj}}\right),\alpha_{-j},\frac{s_{i*}}{s_{ij}}\right)\right)^{s_{ij}} \right\}.$$

The objective function can then be written as

$$\prod_{v\in V_j(s)} \left(u_i(\alpha_j,\alpha_{-j},v) - \sum_{v'\in V_j(s)} p_j(v',s) u_i\left(\alpha_j^d(\alpha_{-j},v'),\alpha_{-j},v\right)\right)^{p_j(v,s)},$$

where $V_j(s) := \left\{v' \in \mathbb{R}^m_+ : \exists i \in N_j(s_{*j}) \text{ with } s_{i*}/s_{ij} = v'\right\}$. Now, take $s,s' \in S$ such that s' is a rearrangement of s for firm j, that is, $p_j(v,s') = p_j(v,s)$ for every $v \in \left\{v' \in \mathbb{R}^m_+ : v'_j = 1\right\}$. Then, given any α_{-j}, the last objective function is the same under s and under s', so the maxima are the same.

Part (ii) Let g_j be WAPP with $\gamma_{*j} = \gamma_{*j}^{sp-0}$, that is, for every $s \in S$ and $\alpha_{-j} \in \times_{k\neq j}\Delta(A_{-k})$

$$\begin{aligned}
g_j(\alpha_{-j},s) &= \operatorname*{arg\,max}_{\alpha_j\in\Delta(A_j)} \left\{ \sum_{i\in N_j(s_{*j})} \frac{1}{|N_j(s_{*j})|} u_i(\alpha_j,\alpha_{-j},s_{i*}) \right\} \\
&= \operatorname*{arg\,max}_{\alpha_j\in\Delta(A_j)} \left\{ \sum_{i\in N_j(s_{*j})} \frac{s_{ij}}{|N_j(s_{*j})|} u_i(\alpha_j,\alpha_{-j},s_{i*}/s_{ij}) \right\} \\
&= \operatorname*{arg\,max}_{\alpha_j\in\Delta(A_j)} \left\{ \sum_{i\in N_j(s_{*j})} s_{ij} u_i(\alpha_j,\alpha_{-j},s_{i*}/s_{ij}) \right\} \\
&= \operatorname*{arg\,max}_{\alpha_j\in\Delta(A_j)} \left\{ \sum_{v\in V_j(s)} p_j(v,s) u_i(\alpha_j,\alpha_{-j},v) \right\},
\end{aligned}$$

where the second equality follows by the linearity of $u_i(\alpha_j,\alpha_{-j},s_{i*})$ in s_{i*} and the third because $1/|N_j(s_{*j})|$ is a positive number that does not vary with i. Thus, $g_j(\alpha_{-j},s)$ depends on s only through the values of $p_j(v,s)$ for different $v \in \{v' \in \mathbb{R}^m_+ : v'_j = 1\}$, and the result follows. **Q.E.D.**

Proof of Claim 1 First notice that

$$\lambda_{12}(n) = \lambda_{21}(n) = \frac{\sigma\gamma_{11}(n)}{\sigma\gamma_{11}(n) + \frac{n-1}{2}\frac{2(1-\sigma)}{n-1}\frac{2(1-\gamma_{11}(n))}{n-1}} = \frac{1}{1 + \frac{2(\sigma^{-1}-1)((\gamma_{11}(n))^{-1}-1)}{n-1}}.$$

(i) $\lambda_{12}(n)$ is increasing in n if and only if the denominator is decreasing in n, or equivalently

$$-\frac{\frac{\partial\gamma_{11}(n)}{\partial n}}{2(\gamma_{11}(n))^2}(n-1) - \left((\gamma_{11}(n))^{-1} - 1\right) < 0.$$

(ii) If $\lim_{n\to\infty} n\gamma_{11}(n) = \infty$, then $\lim_{n\to\infty}[2(\sigma^{-1}-1)((\gamma_{11}(n))^{-1}-1)]/(n-1) = 0$, since the numerator is bounded and the denominator goes to ∞. Thus, $\lim_{n\to\infty}\lambda_{12}(n) = 1$.

(iii) We have that

$$\gamma_{11}(n) = \frac{\sigma^\theta}{\sigma^\theta + \left(\frac{2}{n-1}\right)^{\theta-1}(1-\sigma)^\theta} = \frac{1}{1 + \left(\frac{2}{n-1}\right)^{\theta-1}\left(\frac{1-\sigma}{\sigma}\right)^\theta}, \quad \text{so that}$$

$$\lambda_{12}(n) = \lambda_{21}(n) = \frac{1}{1 + \frac{2(\sigma^{-1}-1)\left(\frac{2}{n-1}\right)^{\theta-1}\left(\frac{1-\sigma}{\sigma}\right)^\theta}{n-1}} = \frac{1}{1 + \left(\frac{2}{n-1}\right)^\theta\left(\frac{1-\sigma}{\sigma}\right)^{1+\theta}}$$

and the rest follows.

(iv) See Dubey and Shapley (1979, p. 112-114).

(v) We have that

$$\gamma_{11}(n) \equiv \gamma_{11}^{mB}(n) = \frac{\gamma_{11}^B(n)/\sigma}{\gamma_{11}^B(n)/\sigma + \frac{n-1}{2}\frac{\frac{2\left(1-\gamma_{11}^B(n)\right)}{n-1}}{\frac{2(1-\sigma)}{n-1}}} = \frac{1}{1 + \frac{n-1}{2}\frac{\left(\gamma_{11}^B(n)\right)^{-1}-1}{\sigma^{-1}-1}} \quad \text{so that}$$

$$\lambda_{12}(n) = \lambda_{21}(n) = \frac{1}{1 + \frac{2(\sigma^{-1}-1)\frac{n-1}{2}\frac{\left(\gamma_{11}^B(n)\right)^{-1}-1}{\sigma^{-1}-1}}{n-1}} = \gamma_{11}^B(n).$$

and the result follows from part (iv). **Q.E.D.**

Proof of Claim 2 is trivial and thus left to the reader.

Proof of Proposition 4 The FOCs in equilibrium give (see section 4.2):

$$P(Q^*) - C_j'(q_j^*) + P'(Q^*)\left[q_j^* + \sum_{k\in M\setminus\{j\}} \lambda_{jk}^*(s) q_k^*\right] = 0,$$

and the result follows. **Q.E.D.**

Proof of Proposition 5 The partial derivative of $f(q,s)$ with respect to β_{ij} is

$$\begin{aligned}\frac{\partial f(q,s)}{\partial \beta_{ij}} &= \left[\frac{\gamma_{ij}^*}{\beta_{ij}}\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j} - \frac{1}{u_i - d_{ij}}\frac{1}{\sum_{h\in N_j(\beta_{*j})}\frac{\beta_{hj}}{u_h - d_{hj}}}\sum_{t\in N_j(\beta_{*j})}\gamma_{tj}^*\frac{\partial u_t\left(q,s_{t*}\right)}{\partial q_j}\right]\cdot \mathbf{e}_j \\ &= \frac{\gamma_{ij}^*}{\beta_{ij}}\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j}\cdot \mathbf{e}_j,\end{aligned}$$

where $\mathbf{e}_j$ the m-dimensional standard unit vector with 1 in its j-th dimension. It follows by the Implicit Function Theorem that

$$\begin{aligned}\begin{pmatrix}\frac{dq_1^*}{d\beta_{ij}}\\ \frac{dq_2^*}{d\beta_{ij}}\\ \vdots \\ \frac{dq_m^*}{d\beta_{ij}}\end{pmatrix} &= -J^{-1}(q^*,s)\left.\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j}\right|_{q=q^*}\cdot \mathbf{e}_j = -\left(\det\left(J\right)\right)^{-1}\frac{\partial u_i}{\partial q_j}\cdot \operatorname{adj}\left(J\right)\mathbf{e}_j \\ &= -\left(\det\left(J\right)\right)^{-1}\frac{\partial u_i}{\partial q_j}\cdot \begin{pmatrix}(-1)^{1+j}\det\left(J_{-j-1}\right)\\ (-1)^{2+j}\det\left(J_{-j-2}\right)\\ \vdots \\ (-1)^{m+j}\det\left(J_{-j-m}\right)\end{pmatrix},\end{aligned}$$

where the second equality follows from the Laplace expansion, $\operatorname{adj}(J)$ is the adjugate or classical adjoint of J, and J_{-j-k} is the J matrix with the j-th row and k-th column removed. Since J is negative definite

$$\operatorname{sgn}\left\{\det\left(J\right)\right\} = -\operatorname{sgn}\left\{\det\left(J_{-j-j}\right)\right\} = \operatorname{sgn}\{(-1)^m\},$$

$$\text{so that}\quad \operatorname{sgn}\left\{\frac{dq_j^*}{d\beta_{ij}}\right\} = \operatorname{sgn}\left\{(-1)^{2j}\frac{\partial u_i}{\partial q_j}\right\} = \operatorname{sgn}\left\{\left.\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j}\right|_{q=q^*}\right\},$$

where

$$\left.\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j}\right|_{q=q^*} = \sum_{h=1}^{m} s_{ih}\left.\frac{\partial \pi_h(q,s_{i*})}{\partial q_j}\right|_{q=q^*} = P(Q^*)\left[s_{ij}\frac{P(Q^*)-C_j'(q_j^*)}{P(Q^*)} - \frac{\sum_{h=1}^{m} s_{ih}w_h^*}{\eta(Q^*)}\right]$$

$$= -Q^*P'(Q^*)\left[s_{ij}\left(w_j^*+\overline{\lambda}_j\right) - \sum_{h=1}^{m} s_{ih}w_h^*\right] = -Q^*P'(Q^*)s_{ij}\left(\overline{\lambda}_j^* - \overline{\lambda}_{i;j}^*\right),$$

and the result follows. **Q.E.D.**

Appendix B: Supplementary results

Lemma 1 provides conditions for assumption (iv) of Proposition 1 to hold.

Lemma 1. Fix an $s \in S$ and let condition (i) of Proposition 1 hold. For each firm $j \in M$ let the corporate control mechanism g_j be $\mathrm{NB}_{\beta_{*j},d_{*j}}$. $B_j^P(a_{-j})$ is lower hemicontinuous in $a_{-j} \in \widetilde{A}_{-j}$ if any of the following three conditions hold.

(i) For every $j \in M$, conditions (ii) and (v) of Proposition 1 hold, and for every $a_{-j} \in \widetilde{A}_{-j}$ there exists $a_j \in A_j$ such that $u(a_j,a_{-j},s) \gg d_{*j}(a_{-j},s)$.

(ii) For every $j \in M$, conditions (ii) and (iii) of Proposition 1 hold and for every $a_{-j} \in \widetilde{A}_{-j}$, $B_j^P(a_{-j},s) \subseteq \mathrm{cl}\left(\{a_j \in A_j : u(a_j,a_{-j},s) \gg d_{*j}(a_{-j},s)\}\right)$.

(iii) For every $j \in M$, $\widetilde{A}_j \subset \mathbb{R}^{r_j}$ is an r_j-dimensional compact and convex polytope.

Proof of Lemma 1 Part (i) follows from Proposition 4.2 in Dutang (2013), which is an application of Theorem 5.9 in Rockafellar and Wets (1997). Part (ii) follows from Proposition 4.3 in Dutang (2013); see also Theorem 13 of Hogan (1973). Part (iii) follows from Corollary 2 in Maćkowiak (2006). A similar result is also given in Claim 2 of Banks and Duggan (2004). **Q.E.D.**

Lemma 2 provides conditions under which in a Cournot oligopoly an investor's portfolio profit is strictly concave in a firm's quantity.

Lemma 2. Fix an investor $i \in N$ and a firm $j \in M$. If for every quantity profile q such that $Q < \overline{Q}$ it holds that

$$E(Q)\sum_{k\in M} s_{ik}w_k < 1 + s_{ij}\left(1 - \frac{C''(q_j)}{P'(Q)}\right),$$

where $E(Q) := -P''(Q)Q/P'(Q)$ the (absolute value of the) elasticity of the slope of inverse demand, then for any q_{-j}, $u_i(q,s_{i*})$ is strictly concave in q_j for every q_j such that $Q < \overline{Q}$. A sufficient condition is

$$E(Q) < \frac{1+s_{ij}}{\max_{k\in M} s_{ik}} \; \forall Q \in \left[0,\overline{Q}\right) \quad \text{and} \quad C''(q_j) \geq 0 \; \forall q_j.$$

Proof of Lemma 2 The derivative of $u_i\left(q_j,q_{-j},s_{i*}\right)$ with respect to q_j is given by

$$\frac{\partial u_i\left(q_j,q_{-j},s_{i*}\right)}{\partial q_j} = s_{ij}\left(P(Q) - C'(q_j)\right) + P'(Q)\sum_{k\in M} s_{ik}q_k,$$

and the second derivative by

$$\begin{aligned}\frac{\partial^2 u_i\left(q_j,q_{-j},s_{i*}\right)}{\partial q_j^2} &= (1+s_{ij})P'(Q) - s_{ij}C''(q_j) + P''(Q)\sum_{k\in M} s_{ik}q_k \\ &= P'(Q)\left[1 + s_{ij}\left(1 - \frac{C''(q_j)}{P'(Q)}\right) - E(Q)\sum_{k\in M} s_{ik}w_k\right],\end{aligned}$$

and the result follows. **Q.E.D.**

Lemma 3 characterizes a firm's problem in a Cournot oligopoly.

Lemma 3. Assume that assumed there exists $\overline{q} > 0$ such that $P(q) < C_j(q)/q$ for every $q > \overline{q}$ and every firm $j \in M$. Fix a firm $j \in M$ and q_{-j} and let the corporate control mechanism g_j be $\mathrm{NB}_{\beta_{*j},d_{*j}}$. Assume that for every investor $i \in N$, $u_i(q,s_{i*})$ is strictly concave in q_j. Then, the following statements are true:

(i) $B_j^P\left(q_{-j},s\right) \coloneqq \{q_j \in A_j : u\left(q_j,q_{-j},s\right) \geq d_{*j}(q_{-j},s)\}$ is a closed interval,

(ii) $g_j\left(q_{-j},s\right)$ is a singleton,

(iii) the Nash product is increasing (resp. decreasing) in q_j for $q_j \overset{(\text{resp.}>)}{<} g_j\left(q_{-j},s\right)$, and

(iv) if $\exists q_j$ such that $d_i(q_{-j},s) < u_i\left(q_j,q_{-j},s_{i*}\right)$ for every $i \in N_j(\beta_{*j})$, then $g_j\left(q{-}j,s\right)$ solves the FOC.

Proof of Lemma 3 Since for $q_j > \overline{q}$ profit becomes negative, we can constrain each firm to choose quantity $q_j \in [0,\overline{q}]$. From continuity of u_i in q_j and the definition of B_j^P it follows then that B_j^P is compact. Especially, given strict concavity of u_i in q_j for every i, it follows that B_j^P is convex, thus a closed interval. We distinguish the following two cases:

Case 1: Given that u_i is strictly concave in q_j for every i (so u_i can be equal to d_{ij} for at most 2 values of q_j in B_j^P), the only way that $\forall q_j \in B_j^P$ there exists $i \in N$ such that $d_{ij}(q_{-j},s) = u_i\left(q_j,q_{-j},s_{i*}\right)$ is for B_j^P to be a singleton. By continuity of u_i in q_j, this means that $d_{ij}(q_{-j},s)$ is equal to $\max_{q_j} u_i\left(q_j,q_{-j},s_{i*}\right)$ for some $i \in N$, and the relevant results follow.

Case 2: If $\exists q_j \in B_j^P$ such that $d_{*j}(q_{-j},s) \ll u\left(q_j,q_{-j},s\right)$, we have that for every $i \in N$ and every $q_j \in B_j^P(q_{-j},s)$

$$\begin{aligned}&\frac{\partial^2\left(u_i\left(q_j,q_{-j},s_{i*}\right)-d_{ij}(q_{-j},s)\right)^{\beta_{ij}(s_{*j})}}{\partial q_j^2}\\ &=-\frac{\beta_{ij}(s_{*j})\left(1-\beta_{ij}(s_{*j})\right)}{\left(u_i\left(q_j,q_{-j},s_{i*}\right)-d_{ij}(q_{-j},s)\right)^{2-\beta_{ij}(s_{*j})}}\left(\frac{\partial u_i\left(q_j,q_{-j},s_{i*}\right)}{\partial q_j}\right)^2\\ &\quad+\frac{\beta_{ij}(s_{*j})}{\left(u_i\left(q_j,q_{-j},s_{i*}\right)-d_{ij}(q_{-j},s)\right)^{1-\beta_{ij}(s_{*j})}}\frac{\partial^2 u_i\left(q_j,q_{-j},s_{i*}\right)}{\partial q_j^2}<0,\end{aligned}$$

by strict concavity of u_i in q_j. Also, for every i, $\left(u_i\left(q_j,q_{-j},s_{i*}\right)-d_{ij}(q_{-j},s)\right)^{\beta_{ij}(s_{*j})}$ is non-negative and not identically equal to zero over B_j^P. The results then follow from Theorem 4 in Kantrowitz and Neumann (2005). **Q.E.D.**

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