Corporate control under common ownership*

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Abstract

I provide an axiomatic analysis of corporate control under common ownership. I show that the current canonical model of corporate control under common ownership imposes two restrictions on firm behavior: (i) that the firm is efficiently controlled, and (ii) that the distribution of power across shareholders within the firm depends only on the firm's ownership structure, and not on external factors such as the stakes of the firm's shareholders in competing firms, the strategies of other firms, or market conditions. I propose the Nash bargaining (NB) model, which models the firm's behavior as the result of asymmetric Nash bargaining among the firm's shareholders. NB generalizes the current canonical model by allowing for external factors to influence the distribution of power across the firm's shareholders. I characterize how additional restrictions on firm behavior constrain the classes of WAPP and NB models. The results guide researchers and practitioners on how to model corporate control under common ownership and how to robustly test the common ownership hypothesis.

Keywords: common ownership, corporate governance, bargaining, Nash bargaining, Nash-in-Nash, minority shareholdings, antitrust, competition policy, oligopoly
JEL classification codes: C71, D43, G34, L11, L13, L21, L41

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1 Introduction

Perfect competition is crucial for shareholders to unanimously agree on own-firm profit maximization (Hart, 1979), which has been the standard assumption on firm behavior at least since Fisher's (1930) separation theorem. Yet, recent work has shown that firm market power has been widespread and increasing in the U.S. economy (Loecker et al., 2020). At the same time, there is evidence that absent perfect competition firms may not seek to maximize own profit. Particularly, if a firm's shareholders also hold shares in competing firms and the firm's manager wants to maximize shareholder value, then she will not maximize the firm's profit. Indeed, common ownership has been argued to induce firms to partially internalize the effect their actions have on competing firms' profits, thus softening competition (see, e.g., Posner et al., 2017; Azar et al., 2018; Schmalz, 2018).

In studying such anti-competitive effects—be it theoretically (see, e.g., Vives and Vravosinos, 2025) or empirically (see, e.g., Backus et al., 2021b), a model of corporate control other than own-profit maximization is often necessary. The model needs to describe how a firm's conduct is shaped by the shareholders' conflicting interests. For example, shareholders with smaller holdings in competing firms will want the firm to price more aggressively than shareholders with larger stakes in other firms.

Modeling corporate control can be more or less complicated depending on the ownership structure. When a unique shareholder owns the majority of a firm's shares, it is reasonable to model that firm as trying to maximize that shareholder's wealth. Indeed, previous works have recognized this and for simplicity assumed each firm to be controlled by a majority shareholder (see, e.g., Antón et al., 2023). However, in practice, most large firms are held by multiple minority shareholders, whose holdings across firms in the same industry vary. It is then not as simple to decide on a satisfying model of corporate control. Most of the literature has so far relied on what I call the *weighted average portfolio profit* model (WAPP) put forward by Rotemberg (1984), Bresnahan and Salop (1986), and O'Brien and Salop (2000). This model poses that, given a set N of n shareholders and a set M of firms in an industry, the manager of firm f maximizes a weighted average of the shareholders' portfolio profits, that is,

$$\sum_{i \in N} \gamma_{if}(s_{*f}) \sum_{k \in M} s_{ig}\pi_g \propto \pi_f + \sum_{g \in M \setminus \{f\}} \overbrace{\frac{\sum_{i \in N} \gamma_{if}(s_{*f}) s_{ig}}{\sum_{i \in N} \gamma_{if}(s_{*f}) s_{if}}}^{=:\lambda_{fg}} \pi_g,$$

where s_{if} is shareholder i's shares with cash-flow rights over firm f's profits, $\gamma_{if}(s_{*f})$ her control weight over firm f, which depends on the ownership structure $s_{*f} \equiv (s_{1f}, s_{2f}, \ldots, s_{nf})$ of the firm, and π_g is firm g's profit. Equivalently, firm f maximizes its own profit plus each other firm g's profit weighted by λ_{fg}, the Edgeworth coefficient of effective sympathy from firm f towards firm g. The literature usually makes the *proportional control* assumption that $\gamma_{*f}(s_{*f}) = s_{*f}$.

However, as multiple authors have recognized, there is limited understanding around modeling corporate control under common ownership (see, e.g., Schmalz, 2018; Backus et al., 2021a; Antón et al., 2023). Although WAPP is simple and instinctively reasonable, it imposes restrictions on firm behavior that merit careful study. Much of the debate has focused on choosing the "correct" mapping γ_{*f} from ownership structure to control weights. Indeed, "any formulation of γ is implicitly a model of corporate governance and one where theory offers precious little guidance" (Backus et al., 2021a). Nevertheless, the assumption that there even *exists* a correct mapping γ_{*f} such that the firm's behavior can be written as the solution to the maximization of the WAPP objective function may itself not be innocuous. Therefore, a theoretical analysis will be useful for (i) studying how properties of firm behavior translate into restrictions on the mapping γ_{*f}, (ii) evaluating the properties of firm behavior that allow it to even admit a WAPP representation, and (iii) developing an alternative model of corporate control under common ownership for when there is a concern that firm behavior may not admit a WAPP representation.

I pursue each of the three objectives. First, within the context of the WAPP model, I propose two monotonicity properties that capture the notion that "more shares should lead to more control." The first property, called *rank preservation*, has to do with how the firm adjusts its strategy in response to changes in its shareholders' interests. Starting from an ownership structure where firm f's shareholders' interests are aligned, consider a stock trade between two shareholders i and j of firm f in which i buys shares of another firm $g \neq f$ from j. The stock trade causes disagreement among firm f's shareholders: shareholder i wants firm f to adjust its strategy in the direction benefiting firm g, while shareholder j wants firm f to adjust its strategy in the opposite direction. Firm f's corporate control mechanism is rank-preserving if, in response to the stock trade, firm f adjusts its strategy in the direction preferred by the larger shareholder involved in the trade. I show that that rank preservation is satisfied if and only if $s_{if} \geq s_{jf} \implies \gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$. The second monotonicity property, *stock-trade monotonicity*, requires that a shareholder i's control power over firm f increases when i grows her stake in every firm by buying shares from another shareholder. I show that stock-trade monotonicity is satisfied if and only if $\gamma_{if}(s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)) \geq \gamma_{if}(s_{*f})$ for any $t > 0$, where $\mathbf{e}_i$ is the standard basis vector with 1 in its i-th entry.

I also characterize a generalization of WAPP with proportional control posing that there exists function δ such that $\gamma_{if}(s_{*f}) = \delta(s_{if})/\sum_{j\in N}\delta(s_{jf})$, which has, for example, been used in Backus et al. (2021a) and Antón et al. (2023). Under proportional control, $\delta(s_{if}) = s_{if}$. I show that a WAPP mechanism admits such a representation if and only if it satisfies three conditions: (i) anonymity, which requires that the identity of shareholders not matter for firm strategy, (ii) inclusivity, which requires that every shareholder of a firm exerts some control over the firm, and (iii) independence of irrelevant shareholders

(IIS), which requires that the relative control power of two shareholders over the firm be not affected by stock trades between other shareholders of the firm. I also discuss when each of these conditions might fail.

Second, I show that two properties are crucial for firm behavior to admit a WAPP representation: *efficiency* and *irrelevance of external factors*. Efficiency requires that for any ownership structure of the firm, there is a subset of shareholders who efficiently control the firm. Namely, firm f never responds to the other firms' strategies by implementing a strategy that could do at least as well for all controlling shareholders and strictly better for at least one of them. Irrelevance of external factors requires that the distribution of power across shareholders within the firm depends only on the firm's ownership structure, and not on external factors such as (i) the stakes of firm f's shareholders in competing firms, (ii) the strategies of other firms, and (iii) market conditions (e.g., demand or production technology).

Nevertheless, it is plausible that in some cases, the distribution of power across a firm's shareholders may depend on such external factors. Firm f's shareholders with larger stakes in other firms in the same industry may pay closer attention to the industry and thus exert more control over firm f. Also, firm f's shareholders who also have shares of competing firm g may have stronger incentives to exert control over firm f when firm g's strategy is such that firm f has a lot of room to influence g's profitability. For instance, if firm g scales up (resp. down) production, firm f's effect through its pricing and production strategy on firm g's profit margin will have a large (resp. small) impact on firm g's profits. Last, firm f's shareholders who also have shares of competing firm g may spend more resources to affect firm f's strategy when firm f and g's goods are strong substitutes, in which case firm f's pricing strategy will have a more pronounced impact on firm g's profits.

Third, I propose the Nash bargaining (NB) model of corporate control under common ownership, a generalization of WAPP which still requires efficient control but allows for external factors to influence the distribution of power across a firm's shareholders. NB models the firm's behavior as the result of asymmetric Nash bargaining among the firm's shareholders. The equilibrium concept is then a Nash equilibrium in Nash bargains. As in WAPP, I study the constraints imposed on NB by the monotonicity, anonymity, inclusivity, and IIS properties. In addition to dispensing with the assumption of irrelevance of external factors, I show that the NB model can relax the tension that arises under WAPP between (i) allowing for atomistic shareholders to collectively exert control over the firm while at the same time (ii) allowing for large shareholders to have control power. As has been noted before (see, e.g., Gramlich and Grundl, 2017; O'Brien and Waehrer, 2017; Brito et al., 2023), I show that generally, under WAPP, as ownership by a group of shareholders with aligned interests is diffused among more and more shareholders, the group of shareholders completely loses any amount of control over firm strategy. For example, as non-common (resp. common) owners becomes dispersed, the firm tends to follow only the common (resp. non-common) owners' interests. While this may be plausible, the possibility that atomistic shareholders may collectively exert control over the firm is also plausible. However, most parametrizations of the WAPP model preclude this possibility. I show that those parametrizations that do allow for this possibility have an unrealistic property: They assign no control power to large shareholders when atomistic shareholders are also present. The NB model allows for the possibility that atomistic shareholders collectively exert control over the firm without imposing this unrealistic property.

After a discussion of related literature, section 2 presents the model. Section 3 characterizes the WAPP and NB models, studies how properties of firm behavior translate into restrictions on the parameters of the firm's objective under WAPP and NB, and discusses the effects of ownership dispersion on firm behavior under WAPP and NB. Section 4 concludes. All proofs are gathered in the Appendix. The Online Appendix provides supplementary results.

Related literature The Nash-in-Nash solution concept has become a standard tool, since it was proposed by Horn and Wolinsky (1988), who study merger incentives when there are exclusive vertical relationships. The current paper fits into the wide literature that has leveraged the Nash-in-Nash solution to study equilibrium outcomes in various environments where the division of surplus between parties (e.g., upstream and downstream firms) plays an important role.[1] It applies Nash-in-Nash to the case of oligopolistic competition among firms when within each firm, shareholders (with varying levels of holdings in competing firms) bargain to decide on firm strategy.

In contrast, theoretical work on corporate control under common ownership has so far focused on microfounding the WAPP mechanism in models of shareholder voting (see, e.g., Azar, 2017; Brito et al., 2018; Moskalev, 2019).[2] Azar and Ribeiro (2022) go a step further modifying the voting model to account for managerial entrenchment, which leads to a generalization of WAPP.[3] They assume that the manager's preference is to maximize her firm's own profit, which implies that relative to WAPP their model is closer to own-profit maximization. Their model also predicts that as ownership becomes dispersed, the manager has more power and thus internalizes the shareholders' interests

[1]For a review of related literature see Collard-Wexler et al. (2019), who also offer a non-cooperative foundation for the solution concept for the case of multiple upstream and downstream firms.

[2]On the other hand, Chiappinelli et al. (2023) consider a setting where shareholders elect managers through the majority rule. Common owners can stir the firm away from own profit towards industry profit maximization by voting for managers that are averse to the negative externality of production. The majority rule implies that large shareholders do not have disproportionately more power than smaller ones.

[3]It can be shown that their model is equivalent to a generalization of WAPP where the manager of firm f is treated as a "virtual" shareholder of the firm with control power γ_f^m and "cash-flow right" s_f^m normalized to $s_f^m = 1$ (so that $s_f^m + \sum_{i \in N} s_{if} = 2$).

to a lesser degree. Although their empirical estimates are qualitatively consistent with this prediction, they show that their voting model overstates this effect. Crucially, it predicts that as ownership becomes infinitely dispersed, the manager tends (in the limit) to maximize own profit—even if all the firm's owners are completely diversified across the industry.

Brito et al. (2023) also try to overcome the shortcomings of WAPP by modifying some of the assumptions in the voting models that microfound WAPP. They argue that under certain assumptions, the resulting weighted average profit weight (WAPW) model does not give excessively more power to larger shareholders. However, I show that WAPW is a reframing of WAPP and even though it indeed gives rise to a parametrization of WAPP that deals with the issue, that parametrization is unrealistic: it gives all shareholders of a firm the same amount of control, so that the firm maximizes the *unweighted* average of its shareholders' portfolio profits.

Apart from overcoming these issues, my approach also differs methodologically from previous works. Instead of microfounding a corporate control model through shareholder voting, I take an axiomatic approach, which allows for more flexibility and avoids the narrow predictions of shareholder voting models. NB mechanisms are characterized as the class of efficient mechanisms and WAPP as a special case of NB. Proportional control is behaviorally defined in terms of a firm's best-response correspondence.

2 Corporate control mechanisms

A tuple $G := \langle N, M, (A_f)_{f\in M}, (\pi_f)_{f\in M}, (s_{if})_{(i,f)\in N\times M}\rangle$ characterizes an oligopoly game with common ownership, where $N := \{1,2,\dots,n\}$ is a set of n shareholders, $M := \{1,2,\dots,m\}$ is a set of m firms, A_f is firm f's strategy space. We will use i,j,k to denote shareholders and f,g,h, to denote firms. Let the strategy profile space be denoted by $A := \times_{f\in M} A_f$. For a strategy profile $a \equiv (a_1,\dots,a_m) \in A$, where $a_f \in A_f$ is firm f's strategy, a_{-f} denotes the profile of strategies of all firms except f, and accordingly $A_{-f} := \times_{g\neq f} A_g$. Firm f's profit function is $\pi_f : A \to \mathbb{R}$, and $s \in S := \{s \in [0,1]^{n\times m} : \sum_{i\in N} s_{if} = 1\ \forall f \in M\}$ is the ownership matrix, where s_{if} denotes shareholder i's share of firm f.[4] This means that i has a cash-flow right over fraction s_{if} of firm f's profits. Shareholder i's total portfolio profit function is $u_i(a,s_{i*}) := \sum_{f\in M} s_{if}\pi_f(a)$.[5] A shareholder i is a shareholder of firm f if $s_{if} > 0$. $N_f(s_{*f}) := \{i \in N : s_{if} > 0\}$ is the set of shareholders of firm f.

A corporate control mechanism $R_f : \times_{g\neq f}\Delta(A_g) \times S \to \Delta(A_f)$ of firm f determines the nonempty set $R_f(\alpha_{-f},s)$ of strategies deemed choosable by firm f for each ownership

[4]Extending the model to allow for short positions (where those shorting a firm's stock do not exert control over the firm) is straightforward. The results follow the same way.

[5]Given a matrix M, M_{i*} and M_{*f} denote M's i-th row and f-th column, respectively. The notation for a function that maps to an $n\times m$ space is analogous.

structure s and each (possibly mixed) strategy profile α_{-f} of the other firms. In principle, the corporate control mechanism should describe firm behavior for any possible profit functions π, given that market conditions such as technology and demand may change. To economize on notation, I suppress this dependence on π. A strategy profile is an equilibrium if every firm plays one of its choosable strategies given the ownership structure and the other firms' strategies.

2.1 The weighted average portfolio profit (WAPP) mechanism

Let Δ^n denote the n-dimensional simplex. I first describe the mechanism of O'Brien and Salop (2000), which I call the weighted average portfolio profit (WAPP) mechanism.

Definition 1. Firm f's corporate control mechanism R_f is a weighted average portfolio profit mechanism if there exists a control power function $\gamma_{*f} : \Delta^n \to \Delta^n$ such that for every $s \in S$ and $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$

(i) the firm maximizes the weighted average portfolio profit of its shareholders:

$$R_f\left(\alpha_{-f}, s\right) = \underset{\alpha_f \in \Delta(A_f)}{\arg\max} \left\{ \sum_{i \in N} \gamma_{if}(s_{*f}) u_i(\alpha_f, \alpha_{-f}, s_{i*}) \right\},$$

(ii) control is exclusive to shareholders: For every $i \in N$, $s_{if} = 0 \implies \gamma_{if}(s_{*f}) = 0$.

This can be rewritten as

$$\underset{\alpha_f}{\arg\max} \left\{ \pi_f(\alpha_f, \alpha_{-f}) + \sum_{g \in M \setminus \{f\}} \overbrace{\frac{\sum_{i \in N} \gamma_{if}(s_{*f}) s_{ig}}{\sum_{i \in N} \gamma_{if}(s_{*f}) s_{if}}}^{=:\lambda_{fg}(s) \geq 0} \pi_g(\alpha_f, \alpha_{-f}) \right\},$$

where $\lambda_{f*}(s)$ is the vector of weights firm f places in firms' profits with λ_{ff} normalized to 1. λ_{fg} is called the Edgeworth (1881) coefficient of effective sympathy of firm f towards firm g. The numerator of λ_{fg} is a measure of the level of cross-holdings of shareholders of firm f in firm $g \neq f$. The denominator measures ownership concentration in firm f.

The literature most commonly assumes that $\gamma_{*f}(s_{*f}) = s_{*f}$, which it calls "proportional control." Backus et al. (2021a) and Antón et al. (2023) consider a generalization of proportional control specifying $\gamma_{if}(s_{*f}) = s_{if}^{\alpha} / \sum_{j \in N} s_{jf}^{\alpha}$ for some $\alpha \geq 0$. $\alpha > 1$ is interpreted as large shareholders having disproportionately more power than smaller shareholders. $\alpha = 1$ corresponds to proportional control. $\alpha < 1$ is interpreted as large shareholders having less than proportionately more power than smaller shareholders. Yet another formulation that has received attention (see, e.g., Azar and Vives, 2022) assumes γ_{*f} to be the normalized Banzhaf power indices of the shareholders (Penrose, 1946; Banzhaf, 1965; Coleman, 1971). To calculate the Banzhaf index, one first enumerates all winning (*i.e.*, with at least 50% of the firm's shares) coalitions of shareholders where there is (at least) one swing shareholder (*i.e.*, a shareholder who is in the coalition and by leaving it would prevent the coalition from reaching majority). The Banzhaf power index of a shareholder is the share of such coalitions where she is a swing shareholder, that is,

$$\gamma_{if}(s_{*f}) = \frac{\left|\left\{T \in 2^N : \sum_{k\in T} s_{kf} \geq 1/2 > \sum_{k\in T\setminus\{i\}} s_{kf}\right\}\right|}{\sum_{t\in N}\left|\left\{T \in 2^N : \sum_{k\in T} s_{kf} \geq 1/2 > \sum_{k\in T\setminus\{t\}} s_{kf}\right\}\right|}.$$

2.2 The Nash bargaining (NB) mechanism

I now describe the Nash bargaining (NB) corporate control mechanism.

Definition 2. Firm f's corporate control mechanism R_f is a Nash bargaining mechanism if there exist a bargaining power function $\beta_{*f} : \Delta^n \to \Delta^n$ and a disagreement payoff function $d_{*f} : \times_{g\neq f}\Delta(A_k) \times S \to \mathbb{R}^n$ such that for every $s \in S$ and $\alpha_{-f} \in \times_{k\neq f}\Delta(A_k)$

(i) the firm maximizes the Nash product:

$$R_f\left(\alpha_{-f},s\right) = \underset{\alpha_f\in B_f(\alpha_{-f},s)}{\arg\max}\left\{\prod_{i\in N_f(\beta_{*f}(s_{*f}))}\left(u_i\left(\alpha_f,\alpha_{-f},s_{i*}\right) - d_{if}(\alpha_{-f},s)\right)^{\beta_{if}(s_{*f})}\right\},$$

where $B_f(\alpha_{-f},s) \coloneqq \{\alpha_f \in \Delta(A_f) : u_i(\alpha_f,\alpha_{-f},s_{i*}) \geq d_{if}(\alpha_{-f},s)\ \forall i \in N_f(\beta_{*f}(s_{*f}))\}$ and $N_f(\beta_{*f}(s_{*f})) \equiv \{i \in N : \beta_{if}(s_{*f}) > 0\}$,

(ii) (control is exclusive to shareholders: For every $i \in N$, $s_{if} = 0 \implies \beta_{if}(s_{*f}) = 0$.

If the maximum Nash product in (i) is positive for every s and α_{-f}, then we say that the NB mechanism has strict benefits from agreement.

3 Properties of corporate control mechanisms

This section discusses several properties of corporate control mechanisms. First, it shows that assuming a mechanism is NB is almost equivalent to requiring that the mechanism satisfy a form of Pareto efficiency. Assuming that a mechanism is WAPP also implies that the mechanism must satisfy this efficiency condition but it also imposes an additional restriction: The distribution of power across shareholders within the firm must be independent of external factors such as (i) market conditions (e.g., market demand or production technology), (ii) the other firms' strategies, and (iii) the other firms' ownership structures. Second, it proposes two monotonicity properties capturing the notion that "more shares should lead to more control," and characterizes when WAPP and NB mechanisms satisfy those properties. Third, it characterizes a class of WAPP mechanisms where $\gamma_{if}(s_{*f}) = \delta(s_{if})/\sum_{j\in N}\delta(s_{jf})$ for some real function δ, as in Backus et al. (2021a) and Antón et al. (2023). Last, it discusses the effects of ownership dispersion under WAPP and NB.

3.1 Efficiency and internal consistency

A corporate control mechanism is efficient if under any ownership structure, there is a subset $\widetilde{N}(s_{*f})$ of the shareholders of firm f who efficiently control the firm. Strong efficiency requires that for any strategy profile of the other firms, firm f never responds by implementing a strategy that is weakly Pareto dominated in the sense that another strategy could do at least as well for all controlling shareholders and strictly better for at least one of them. Weak efficiency requires that for any strategy profile of the other firms, firm f never responds by choosing a strategy such that another strategy could do strictly better for all controlling shareholders.

Definition 3. The corporate control mechanism R_f of firm f is strongly (resp. weakly) efficient if there exists correspondence $\widetilde{N} : \Delta^n \rightrightarrows N$ such that for every $s \in S$,

(i) a nonempty set of shareholders control the firm: $\widetilde{N}(s_{*f}) \neq \emptyset$,

(ii) control is exclusive to shareholders: For every $i \in N$, $s_{if} = 0 \implies i \notin \widetilde{N}(s_{*f})$, and

(iii) the firm is efficiently controlled: For every $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$, there does not exist $\alpha'_f \in \Delta(A_f)$ and $\alpha_f \in R_f(\alpha_{-f}, s)$ such that $u_i(\alpha'_f, \alpha_{-f}, s_{i*}) \geq u_i(\alpha_f, \alpha_{-f}, s_{i*})$ for all $i \in \widetilde{N}(s_{*f})$ with at least one (resp. every) inequality strict.

Firm f's corporate control mechanism R_f is internally consistent if, in addition, for every $s \in S$, $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$, $\alpha_f, \alpha'_f \in R_f(\alpha_{-f}, s)$, and $\alpha''_f \in \Delta(A_f)$

(iv) $u_i(\alpha_f, \alpha_{-f}, s_{i*}) = u_i(\alpha'_f, \alpha_{-f}, s_{i*})$ for all $i \in \widetilde{N}(s_{*f})$, and

(v) if $u_i(\alpha_f, \alpha_{-f}, s_{i*}) = u_i(\alpha''_f, \alpha_{-f}, s_{i*})$ for all $i \in \widetilde{N}(s_{*f})$, then $\alpha''_f \in R_f(\alpha_{-f}, s)$.

Internal consistency requires that the firm's mechanism prescribes strategies that are unique up to payoff-equivalent strategies. With Pareto-dominated strategies already ruled out by efficiency, internal consistency requires that firm f's controlling shareholders not be willing to agree to two different strategies α_f and α'_f when α_f is strictly preferred to α'_f by one controlling shareholder and α'_f is strictly preferred to α_f by another shareholder. In that case, it does not make sense that each of the two controlling shareholders is willing to agree to both policies. Also, if f's controlling shareholders are willing to agree to strategy α_f, then they are also willing to agree to any other strategy that delivers the same portfolio profit to each one of them.

Let $\mathcal{U}_f(\alpha_{-f}, s) := \{v \in \mathbb{R}^{|N_f(s_{*f})|} : \exists \alpha_f \in \Delta(A_f)$ such that $u_j(\alpha_f, \alpha_{-f}, s_{j*}) = v_j$ for every $j \in N_f(s_{*f})\}$ denote the (convex) portfolio profit possibility set of the shareholders of firm f when the other firms' strategy profile is α_{-f}. Proposition 1 studies the efficiency and internal consistency properties of WAPP and NB mechanisms.

Proposition 1. Let firm f's corporate control mechanism be R_f.

(i) If R_f is WAPP, then it is strongly efficient.

(ii) If R_f is NB, then it is weakly efficient.

(iii) If R_f is NB with strict benefits from agreement, then it is strongly efficient and internally consistent.

(iv) If R_f is weakly efficient and internally consistent, then it is NB.

(v) Assume that $\mathcal{U}_f(\alpha_{-f}, s)$ is strictly convex for every α_{-f} and s. If R_f is WAPP or NB, then it is strongly efficient and internally consistent.

Parts (ii)-(iv) show that assuming a corporate control mechanisms is NB is approximately equivalent to assuming it is efficient and internally consistent. When $\mathcal{U}_f(\alpha_{-f}, s)$ is strictly convex for every α_{-f} and s, the class of NB mechanisms coincides with the class of strongly efficient and internally consistent mechanisms, which is a superset of WAPP mechanisms.[6]

3.2 (Ir)relevance of external factors

Any model of corporate control must capture the fact that the distribution of power across a firm's shareholders depends on factors *internal* to firm f. Particularly, the number of shares held be each shareholder must play a primary role: Larger shareholders can be expected to have greater power in shaping firm conduct than smaller ones. Indeed, NB and WAPP capture this.

At the same time, the distribution of power across a firm's shareholders may also depend on *external* factors such as (i) the stakes of firm f's shareholders in competing firms, (ii) the strategies of other firms, and (iii) market conditions (e.g., demand or production technology). First, firm f shareholders with larger stakes in other firms in the same industry may pay closer attention to the industry and thus exert more control over firm f given that firm f's strategy affects the shareholders' portfolio returns not only through its effect on its own profitability but also through its impact on other competing firms' profits. Indeed, there is theoretical and empirical support that investors pay closer attention to a stock when that stock is a larger part of their portfolios (Van Nieuwerburgh and Veldkamp, 2010; Fich et al., 2015; Iliev et al., 2021). Therefore, we can expect investors with stakes in multiple firms in the industry to pay closer attention to the industry. Second, firm f's shareholders who also have shares of competing firm g may have stronger incentives to exert control over firm f when firm g's strategy is such that

[6] The only case where a mechanism can be WAPP but not NB is when $\mathcal{U}_f(\alpha_{-f}, s)$ is only weakly convex, and the mechanism chooses all strategies that result in portfolio profit profiles of firm f's controlling shareholders across a linear segment of the boundary of $\{v \in \mathbb{R}^{|\widetilde{N}(s_{*f})|} : \exists \alpha_f \in \Delta(A_f) \text{ such that } u_j(\alpha_f, \alpha_{-f}, s_{j*}) = v_j \text{ for every } j \in \widetilde{N}(s_{*f})\}$; an NB mechanism cannot choose multiple of them at the same time.

firm f has a lot of room to influence g's profitability. For instance, if firm g expands its production capacity and orders large input quantities to scale up production, firm f's effect through its pricing and production strategy on firm g's profit margin will have a large impact on firm g's profits. On the other hand, if firm g scales back production or even exits a market that firm f operates in, firm f has limited room to affect firm g's profitability. Third, firm f's shareholders who also have shares of competing firm g may spend more resources to affect firm f's strategy when firm f and g's goods are strong substitutes (or, for that matter, complements), since in that case firm f's pricing strategy will have a more pronounced impact on firm g's profits.

WAPP precludes such dependence of the distribution $\gamma_{*f}(s_{*f})$ of power across a firm's shareholders on *external* factors, while NB allows for it. To see this, let A_f be a subset of a Euclidean space with $R_f(\alpha_{-f},s)$ pinned down by the first order condition (FOC). The FOC under WAPP is $\sum_{i\in N}\gamma_{if}(s_{*f})\partial u_i(a_f,a_{-f},s_{i*})/\partial a_f|_{a_f=R_f(\alpha_{-f},s)}=\mathbf{0}$, where $\partial u_i(a_f,a_{-f},s_{i*})/\partial a_f$ is the gradient with respect to a_f. Under NB, the FOC is $\sum_{i\in N_f(\beta_{*f})}\widetilde{\gamma}_{if}(a_{-f},s)\partial u_i(a_f,a_{-f},s_{i*})/\partial a_f|_{a_f=R_f(\alpha_{-f},s)}=\mathbf{0}$ provided $u_i(R_f(\alpha_{-f},s),a_{-f},s_{i*})>d_{if}(a_{-f},s)$ for every $i\in N_f(\beta_{*f})$, where

$$\widetilde{\gamma}_{if}(\alpha_{-f},s):=\frac{\frac{\beta_{if}(s_{*f})}{u_i\left(R_f(\alpha_{-f},s),\alpha_{-f},s_{i*}\right)-d_{if}(\alpha_{-f},s)}}{\sum_{j\in N_f(\beta_{*f})}\frac{\beta_{jf}(s)}{u_j\left(R_f(\alpha_{-f},s),\alpha_{-f},s_{j*}\right)-d_{jf}(\alpha_{-f},s)}}$$

is the disagreement-adjusted control power of shareholder i over firm f at (a_{-f},s). It measures shareholder control accounting for the fact that the further $u_i\,(\alpha_f,\alpha_{-f},s_{i*})$ is from $d_{if}(\alpha_{-f},s)$, the more shareholder i has to lose in case of disagreement. Equivalently, we can write

$$\left.\frac{\partial\pi_f(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(\alpha_{-f},s)}+\sum_{g\in M\setminus\{f\}}\widetilde{\lambda}_{fg}(\alpha_{-f},s)\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(\alpha_{-f},s)}=\mathbf{0},$$

where $\widetilde{\lambda}_{fg}(a_{-f},s):=\sum_{i\in N_f(\beta_{*f})}\widetilde{\gamma}_{if}(a_{-f},s)s_{ig}/\sum_{i\in N_f(\beta_{*f})}\widetilde{\gamma}_{if}(a_{-f},s)s_{if}$ is the weight firm f places on firm g's profit. NB allows for $\widetilde{\gamma}_{*f}(\alpha_{-f},s)$ to depend on the other firms' strategies or ownership structures and on market conditions, since firm f's strategy in case of disagreement can naturally depend on those. For example, if all firms product a homogenous good, and the other firms drive the price lower than firm f's marginal cost, then there is essentially no disagreement and d_{*j} will reflect that the firm should not produce at all. On the other hand, if the other firms keep the price well above firm f's marginal cost, firm f may produce in case of disagreement.

In fact, we can show that the irrelevance of external factors for the distribution of power across shareholders within the firm is what characterizes WAPP as a special case of NB. To see this, define generalized weighted average portfolio profit mechanisms (GWAPP) as follows.

Definition 4. Firm f's corporate control mechanism R_f is a generalized weighted average portfolio profit mechanism (GWAPP) if there exists a control power function $\gamma_{*f} : \times_{g \neq f} \Delta(A_g) \times S \to \Delta^n$ such that for every $s \in S$ and $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$

(i) the firm maximizes the weighted average portfolio profit of its shareholders:

$$R_f\left(\alpha_{-f}, s\right) \subseteq \underset{\alpha_f \in \Delta(A_f)}{\arg\max} \left\{ \sum_{i \in N} \gamma_{if}(\alpha_{-f}, s) u_i(\alpha_f, \alpha_{-f}, s_{i*}) \right\},$$

(ii) control is exclusive to shareholders: For every $i \in N$, $s_{if} = 0 \implies \gamma_{if}(\alpha_{-f}, s) = 0$.

Proposition 2 shows that GWAPP mechanisms are approximately equivalent to NB mechanisms.

Proposition 2. Let firm f's corporate control mechanism be R_f.

(i) If R_f is GWAPP and internally consistent, then it is NB.

(ii) Assume that $\mathcal{U}_f\left(\alpha_{-f}, s\right)$ is strictly convex for every α_{-f} and s. Then, R_f is GWAPP if and only if it is NB.

Therefore, restricting attention to WAPP mechanisms essentially restricts attention to the NB mechanisms which, when expressed as GWAPP mechanisms, have $\gamma_{*f}(\alpha_{-f}, s)$ independent of α_{-f} and s_{*g} for $g \neq f$. An advantage of the NB formulation over GWAPP is that it can be more natural to specify firm f's strategy in case of disagreement—and thereby indirectly specify how α_{-f}, s, and π affect γ's. Within the GWAPP framework, it is hard to directly specify the mapping from α_{-f}, s, and π to γ's.

3.3 Monotonicity

In this section, we characterize monotone corporate control mechanisms, capturing the notion that "the more shares a shareholder has, the more control she exerts over the firm." In doing so, we consider the firm's behavior in the following simple setting. Let A_f be an interval and, whether R_f is WAPP or NB, assume that for every a_{-f} and s, $R_f(a_{-f}, s)$ is pinned down by the FOC with the second-order condition holding strictly. Also assume that and in the case of NB, for every a_{-f} and s, $d_{*f}(a_{-f}, s) = u(a_f, a_{-f}, s)$ for some $a_f \in A_f$. These assumptions need not be understood as limiting the scope of the results. Given that the firm's corporate control mechanism needs to specify the firm's behavior across a range of environments, if it is monotone, it must satisfy the conditions derived in this simple setting.

Before proceeding, we need to develop a language to talk about stock trades. For every shareholder $i \in N_f(s_{*f})$ of firm f, $(\lambda_{i;f1}, \lambda_{i;f2}, \cdots, \lambda_{i;fm}) \equiv \lambda_{i;f*} := \frac{1}{s_{if}} s_{i*} \equiv$

$(s_{i1}/s_{if}, s_{i2}/s_{if}, \ldots, s_{im}/s_{if})$, is the vector of weights i wants firm f to place on firms' profits with the weight on firm f's profit normalized to 1.

Definition 5. Firm f's shareholders unanimously agree on firm conduct under ownership matrix s if $\lambda_{i;f*} = \lambda_{j;f*}$ for every $i,j \in N_f(s_{*f})$. Then, s is called f-unanimous.

We will see that studying the firm's corporate control mechanism locally, around f-unanimous matrices, is a powerful approach. Starting from an f-unanimous matrix, a small stock trade where some firm f shareholders trade firm $g \neq f$ shares can cause firm f to adjust its strategy only through its effect on firm f's shareholders' interests. Even if the stock trade affects the distribution of power across firm f's shareholders (which is possible if firm f's corporate control mechanism is NB), this will not play a role in how firm f adjusts its strategy in response to the stock trade: Given that firm f's shareholders unanimously agree on firm strategy to begin with, the distribution of power among them does not matter.

Definition 6. A $(\psi,g,i,\widetilde{N})$-stock trade is an infinitesimal change in the ownership structure matrix s in direction[7]

$$ds = \left((1-\psi)\mathbf{e}_i - \psi \sum_{j \in \widetilde{N}} \mathbf{e}_j \right) \otimes \mathbf{e}_g,$$

where $\psi \in [0,1]$, $g \in M$, $i \in N \setminus \widetilde{N}$, and $\emptyset \neq \widetilde{N} \subseteq N$.

In a $(\psi,g,i,\widetilde{N})$-stock trade, shareholder i buys firm g shares at rate $1-\psi$, and each shareholder in group $\widetilde{N}$ of shareholders sells firm g shares at rate ψ. When $\widetilde{N} = \{j\}$ is a singleton, we call it a (ψ,g,i,j)-stock trade. In a $(1/2,g,i,j)$-stock trade, shareholder i buys firm g shares from shareholder j.

3.3.1 Rank preservation

We are now ready to study the first monotonicity property: *rank preservation.*

Definition 7. Firm f's corporate control mechanism is rank-preserving if for any firm $g \neq f$, any strategy profile a_{-f} of the other firms, any f-unanimous ownership matrix

[7] In principle, this is possible only when $\psi|\widetilde{N}| = 1-\psi$ or equivalently $\psi = (|\widetilde{N}|+1)^{-1}$, so that ds points inside $S \equiv \{s \in [0,1]^{n\times m} : \sum_{k\in N} s_{kh} = 1 \text{ for every } h \in M\}$. However, to simplify notation, we assume here that there are some additional investors outside the set N who are not firm f shareholders. When $\psi < (|\widetilde{N}|+1)^{-1}$, these additional shareholders sell firm g shares to shareholder i at rate $1-\psi(|\widetilde{N}|+1)$. When $\psi > (|\widetilde{N}|+1)^{-1}$, these additional shareholders buy firm g shares from group $\widetilde{N}$ of shareholders at rate $\psi(|\widetilde{N}|+1)-1$. Similarly, we could allow the number of investors n to vary and require that $R_f(a_{-f},s) = R_f(a_{-f},s')$ for every s,s' such that $N_f(s_{*f}) = N_f(s'_{*f})$ and $s_{j*} = s'_{j*}$ for every shareholder $j \in N_f(s_{*f})$. Then, we could study the effect of a stock trade on $R_f(a_{-f},s)$ by studying its effect on $R_f(a_{-f},s')$ with s's such that $N_f(s_{*f}) = N_f(s'_{*f})$ and $s_{j*} = s'_{j*}$. This would deliver the same results but complicate notation.

s, and any pair of distinct shareholders $i,j \in N_f(s_{*f})$, if $s_{if} \geq s_{jf}$, then the change $\nabla_{ds} R_f(a_{-f},s)$ in firm f's conduct in response to a $(1/2,g,i,j)$-stock trade (which changes the ownership matrix in direction ds) satisfies

$$\left.\frac{\partial \pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f = R_f(a_{-f},s)} \overset{\text{(resp. } \leq)}{\geq} 0 \implies \nabla_{ds} R_f(a_{-f},s) \overset{\text{(resp. } \leq)}{\geq} 0.$$

Starting from an f-unanimous ownership structure, consider a stock trade between two shareholders i and j of firm f in which i buys shares of another firm $g \neq f$ from j. Before the stock trade, firm f's strategy maximizes the portfolio profit of each of its shareholders (given the other firms' strategies). The stock trade causes disagreement among firm f's shareholders regarding firm f's strategy: shareholder i wants firm f to adjust its strategy in the direction benefiting firm g, while shareholder j wants firm f to adjust its strategy in the opposite direction. For example, under Bertrand competition with differentiated products, i will want firm f to price less aggressively while j will want it to price more aggressively after the stock trade. Firm f's corporate control mechanism is rank-preserving if, in response to the stock trade, firm f adjusts its strategy in the direction preferred by the larger shareholder involved in the trade.

Proposition 3 characterizes rank-preserving mechanisms.

Proposition 3. Consider a firm $f \in M$ and assume that for every s, there exist firm $g \neq f$ and $a_{-f} \in A_{-f}$ such that, evaluated at $a_f = R_f(a_{-f},s)$, $\partial \pi_g(a_f,a_{-f})/\partial a_f \neq 0$.

(i) Assume that R_f is WAPP. Then, R_f is rank-preserving if and only if for every s and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$.

(ii) Assume that R_f is NB. Then, R_f is rank-preserving if and only if for every s and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \beta_{if}(s_{*f})/s_{if} \geq \beta_{jf}(s_{*f})/s_{jf}$.

If firm f's mechanism is WAPP, it is rank-preserving if and only if the control power function γ_{*f} preserves the ranking of shareholders in terms of the number of firm f shares they hold. One may instinctively think that under NB, $s_{if} \geq s_{jf}$ implying $\beta_{if}(s_{*f}) \geq \beta_{jf}(s_{*f})$ would be sufficient to capture the idea that larger shareholders exercise more control. However, this is not the case. Under NB, the role of γ_{if} for this result is assumed by β_{if}/s_{if}, *not* β_{if}. The division by s_{if} captures the fact that larger shareholders have more to lose in case of disagreement. Therefore, larger shareholders have more control over firm f than smaller ones if and only if their β's more than compensate for the fact that they have more to lose. For example, if $\beta_{if}(s_{*f}) = s_{if}$ for every shareholder i, then every shareholder has the same control power over firm f in terms of how firm f's strategy changes in response to stock trades around an f-unanimous ownership matrix.

3.3.2 Stock-trade monotonicity

Before defining the second monotonicity property, *stock-trade monotonicity*, we need to define f-neutral stock trades. An f-neutral stock trade does not make firm f want to change its strategy.

Definition 8. Fix an f-unanimous ownership matrix s. A $(\psi,g,i,\widetilde{N})$-stock trade is f-neutral if for any strategy profile a_{-f} of the other firms, firm f's conduct does not change in response to the stock trade (which changes the ownership matrix in direction ds), that is, $\nabla_{ds} R_f(a_{-f},s) = 0$.

Lemma 1 characterizes two types of f-neutral stock trades: (i) those where a firm f shareholder buys firm g shares and every other firm f shareholder sells firm g shares and (ii) those where a firm f shareholder buys firm g shares and another firm f shareholder sells firm g shares.

Lemma 1. Fix an ownership matrix s, and assume that there exists $a_{-f} \in A_{-f}$ such that, evaluated at $a_f = R_f(a_{-f},s)$, $\partial\pi_g(a_f,a_{-f})/\partial a_f \neq 0$.

(i) Assume that R_f is WAPP. Then,

  (a) a $(\psi,g,i,N_f(s_{*f}) \setminus \{i\})$-stock trade is f-neutral if and only if $\gamma_{if}(s_{*f}) = \psi$, and

  (b) a (ψ,g,i,j)-stock trade is f-neutral if and only if $\gamma_{if}(s_{*f})(1-\psi) = \gamma_{jf}(s_{*f})\psi$,

(ii) Assume that s is f-unanimous and R_f is NB. Then,

  (a) a $(\psi,g,i,N_f(s_{*f}) \setminus \{i\})$-stock trade is f-neutral if and only if

$$\frac{\beta_{if}(s_{*f})/s_{if}}{\sum_{j\in N_f(s_{*f})} \beta_{jf}(s_{*f})/s_{jf}} = \psi, \text{ and}$$

  (b) a (ψ,g,i,j)-stock trade is f-neutral if and only if

$$\frac{\beta_{if}(s_{*f})}{s_{if}}(1-\psi) = \frac{\beta_{jf}(s_{*f})}{s_{jf}}\psi.$$

Under WAPP, γ_{if} captures shareholder i's control power over firm f in the following way: If shareholder i's stakes in firm g increase at rate $(1-\gamma_{if})$, while the stake in firm g of every other shareholder of firm f decreases at rate γ_{if}, firm f's conduct does not change. When γ_{if} is high, the other shareholders increase their stakes in firm g by a lot, which should push firm f to adjust its conduct to boost firm g's profits; for example, if f and g produce substitute goods and compete in prices, it should push firm f to increase the price of its good. However, a small decrease in shareholder i's stakes in firm g counteracts this effect, leaving firm f's conduct unchanged. This means that shareholder i exercises a lot of control over firm f. Similarly, $\beta_{if}/s_{if}/(\sum_{j\in N_f(s_{*f})}\beta_{jf}/s_{jf})$ captures shareholder i's control power over firm f under NB. As mentioned before, the division by s_{if} captures the fact that larger shareholders have more to lose in case of disagreement. Similarly, under WAPP, the ratio γ_{if}/γ_{jf} captures shareholder i's control power over firm f relative to shareholder j's control power over it. Under NB, the relevant ratio is $\beta_{if}/s_{if}/(\beta_{jf}/s_{jf})$.

Definition 9. Firm f's corporate control mechanism has stock-trade monotone control if for any firm $g \neq f$, any pair of shareholders $i,j \in N$, any f-unanimous ownership matrix s, any $t \in [0, \min_{g\in M: s_{ig}>0} \min\{1 - s_{ig}/s_{ig}, s_{jg}/s_{ig}\}]$,[8] and any $\psi,\psi' \in [0,1]$, if

(i) starting from s, a $(\psi,g,i,N_f(s_{*f}) \setminus \{i\})$-stock trade is f-neutral, and

(ii) starting from s', a $(\psi',g,i,N_f(s'_{*f}) \setminus \{i\})$-stock trade is f-neutral,

then $\psi' \geq \psi$, where $s'_{k*} = s_{k*}$ for every $k \neq i,j$, $s'_{i*} = (1+t)s_{i*}$, and $s'_{j*} = s_{j*} - ts_{i*}$.

Starting from an f-unanimous ownership structure, consider a stock trade where shareholder i grows her stake in *every* firm by a factor of t by buying shares from shareholder j. Firm f's corporate control mechanism is stock-trade monotone if each shareholder i's control power over firm f—as measured through an f-neutral stock trade between her and every other shareholder of firm f—increases after i grows her stake in every firm by buying shares from shareholder j. Stock-trade monotonicity is consistent with the idea that the more firm f shares shareholder i has, the more control she exerts over firm f. It is also consistent with the possibility that external factors influence the distribution of power across firm f's shareholders, as discussed in the previous section. Particularly, it is compatible with the idea that the more shares i has of *other* firms in the industry, the closer attention she will pay to the industry and, thus, the more influence she will have over firm f's strategy.

Proposition 4 characterizes stock-trade monotone mechanisms.

Proposition 4. Consider a firm $f \in M$ and assume that for every s, there exist firm $g \neq f$ and $a_{-f} \in A_{-f}$ such that, evaluated at $a_f = R_f(a_{-f},s)$, $\partial\pi_g(a_f,a_{-f})/\partial a_f \neq 0$.

(i) Assume that R_f is WAPP. Then, R_f has stock-trade monotone control if and only if for every s, every pair of firm f shareholders $i,j \in N_f(s_{*f})$, and every $t \in [0, \min\{s_{jf}, 1 - s_{if}\}]$, $\gamma_{if}(s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)) \geq \gamma_{if}(s_{*f})$.

(ii) Assume that R_f is NB. Then, R_f has stock-trade monotone control if and only if for every s, every pair of firm f shareholders $i,j \in N_f(s_{*f})$, and every $t \in [0, \min\{s_{jf}, 1 - s_{if}\}]$,

$$\frac{\beta_{if}(s'_{*f})/s'_{if}}{\sum_{k\in N_f(s_{*f})}\beta_{kf}(s'_{*f})/s'_{kf}} \geq \frac{\beta_{if}(s_{*f})/s_{if}}{\sum_{k\in N_f(s_{*f})}\beta_{kf}(s_{*f})/s_{kf}},$$

[8] The constraint on t guarantees that $s'_{i*} \leq 1$ and $s'_{j*} \geq 0$.

where $s'_{*f} \coloneqq s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)$.

3.4 Independence of irrelevant shareholders

In this section, I constrain s to lie in $\{s \in S : |N_f(s_{*f})| \geq 3\}$. I characterize the generalization of WAPP with proportional control posing that there exists δ such that $\gamma_{if}(s_{*f}) = \delta(s_{if})/\sum_{j\in N}\delta(s_{jf})$ is the control power function. This formulation of WAPP has, for example, been used in Backus et al. (2021a) and Antón et al. (2023). Proposition 5 shows that a WAPP mechanism admits such a representation if and only if it satisfies three conditions.

The first condition is anonymity, which requires that the identity of shareholders not matter for firm strategy. Namely, permuting the ownership matrix s does not change the firm's corporate control mechanism. For example, if all shares of shareholder i are transferred to shareholder j, and all shares of shareholder j are transferred to shareholder i, the firm will choose the same strategy as it did before under every scenario.

Definition 10. A permutation matrix is an $n \times n$ matrix that has exactly one entry of 1 in each row and each column with all other entries 0.

Definition 11. Firm f's corporate control mechanism is anonymous if for any s, a_{-f}, and permutation matrix P, $R(a_{-f},s) = R(a_{-f},Ps)$.

The second condition is inclusivity, which requires that every shareholder of a firm exerts some control over the firm.

Definition 12. Firm f's corporate control mechanism is inclusive if for any firm $g \neq f$, any f-unanimous ownership matrix s, any pair of firm f's shareholders $i,j \in N_f(s_{*f})$, a $(0,g,i,j)$-stock trade is not f-neutral.[9]

The third condition is independence of irrelevant shareholders (IIS), which requires that the relative control power of two shareholders over firm f be not affected by stock trades between other shareholders of the firm.

Definition 13. Firm f's corporate control mechanism satisfies independence of irrelevant shareholders (IIS) if for any firm $g \neq f$, any f-unanimous ownership matrices s and s' such that $s'_{if} = s_{if}$ and $s'_{jf} = s_{jf}$, any pair of shareholders $i,j \in N_f(s_{*f})$, and any $\psi \in [0,1]$, if a (ψ,g,i,j)-stock trade is f-neutral starting from s, then it is f-neutral also starting from s'.

Proposition 5. Consider a firm $f \in M$ and assume that for every s, there exist firm $g \neq f$ and $a_{-f} \in A_{-f}$ such that, evaluated at $a_f = R_f(a_{-f},s)$, $\partial\pi_g(a_f,a_{-f})/\partial a_f \neq 0$. Assume that R_f is WAPP. Then, R_f satisfies anonymity, inclusivity, and IIS if and only if there exists non-decreasing $\delta : [0,1] \to \mathbb{R}_+$ with $\delta(0) = 0$ and $\delta(x) > 0$ for every $x > 0$ such that for every s and every $i \in N$, $\gamma_{if}(s_{*f}) = \delta(s_{if})/\sum_{j\in N}\delta(s_{jf})$.

[9] After relabeling of the shareholders, this also implies that a $(1,g,i,j)$-stock trade is not f-neutral either.

Clearly, there can be objections to each of the three conditions. Anonymity may fail if different investors have different expertise or power to influence firm strategy. For example, a venture capitalist may influence firm strategy more than an individual investor. Inclusivity will fail if there are fixed costs in exerting control over the firm; for instance, it can be too costly to follow industry developments and vote for small shareholders. Last, to see how IIS may fail, consider the following scenario: s_{*f} changes from (0.1,0.2,0.35,0.35) to (0.1,0.2,0.45,0.25) as shareholder 3 buys shares from shareholder 4. Before the stock trade, shareholder 1 is not needed to reach majority in shareholder voting, while shareholder 2 can help each of shareholders 3 and 4 to reach majority. Therefore, before the stock trade, it is plausible that shareholder 2 has much more power than shareholder 1. After the stock trade, shareholder 1 is needed in coalitions (1,3) and (1,2,4), and shareholder 2 is needed in coalitions (2,3) and (1,2,4). Thus, the power balance between shareholders 1 and 2 may become more equal after the stock trade between shareholders 3 and 4.

3.5 Powerlessness of diffuse ownership

First, it shows that NB is more flexible than WAPP in accounting for ownership dispersion.

Definition 14. Firm f's shareholders are divided in their preferences on firm conduct under ownership matrix s if there exists a partition $\{N_1,N_2\}$ of $N_f(s_{*f})$ such that for every $i,j \in N_f(s_{*f})$, if (i) $i,j \in N_1$ or $i,j \in N_2$, then $\lambda_{i;f*} \equiv s_{i*}/s_{if} = s_{j*}/s_{jf} \equiv \lambda_{j;f*}$ for every $i,j \in N_f(s_{*f})$, while (ii) if $i \in N_1$ and $j \in N_2$, then $\lambda_{i;f*} \neq \lambda_{j;f*}$. Then, s is called f-bianimous.

Ownership diffusion under WAPP. Fix some $i_1 \in N_1$ and $i_2 \in N_2$. For any firm $g \neq f$,

$$\begin{aligned}\lambda_{fg}(s) &= \frac{\sum_{i\in N_1}\gamma_{if}(s_{*f})s_{ig} + \sum_{i\in N_2}\gamma_{if}(s_{*f})s_{ig}}{\sum_{i\in N_1}\gamma_{if}(s_{*f})s_{if} + \sum_{i\in N_2}\gamma_{if}(s_{*f})s_{if}}\\ &= \frac{\lambda_{i_1;fg}\sum_{i\in N_1}\gamma_{if}(s_{*f})s_{if} + \lambda_{i_2;fg}\sum_{i\in N_2}\gamma_{if}(s_{*f})s_{if}}{\sum_{i\in N_1}\gamma_{if}(s_{*f})s_{if} + \sum_{i\in N_2}\gamma_{if}(s_{*f})s_{if}}\\ &= \frac{\lambda_{i_2;fg} + \lambda_{i_1;fg}\frac{\sum_{i\in N_1}\gamma_{if}(s_{*f})s_{if}}{\sum_{i\in N_2}\gamma_{if}(s_{*f})s_{if}}}{1 + \frac{\sum_{i\in N_1}\gamma_{if}(s_{*f})s_{if}}{\sum_{i\in N_2}\gamma_{if}(s_{*f})s_{if}}}.\end{aligned}$$

Consider a sequence $s(\nu)_{\nu\in\mathbb{N}}$ of f-bianimous ownership matrices such that $\lambda_{i_1;fg}$, $\lambda_{i_2;fg}$, N_1, and s_{i*} are fixed along the sequence for every $i \in N_1$, but the holdings of shareholders in N_2 are divided across more and more shareholders, so that $|N_2| \to \infty$ and $\max_{i\in N_2} s_{if} \to 0$. Then, $\sum_{i\in N_2}\gamma_{if}(s_{*f})s_{if} \to 0$, so $\lambda_{fg}(s) \to \lambda_{i_1;fg}$ unless $\sum_{i\in N_1}\gamma_{if}(s_{*f})s_{if} \to 0$ at the same or faster rate than $\sum_{i\in N_2}\gamma_{if}(s_{*f})s_{if} \to 0$. That is, the weight firm f assigns on firm g's profit converges to the weight group N_1 of shareholders assigns to it. If $\lambda_{fg}(s) \to \lambda_{i_1;fg}$

for every firm $g \neq f$, then, under standard assumptions, firm f's strategy will converge to the strategy most preferred by the group N_1 of shareholders.[10] For $\sum_{i \in N_2} \gamma_{if}(s_{*f}) s_{if} \to 0$, it is necessary that $\max_{i \in N_1} \gamma_{if}(s_{*f}) \to 0$. In that case, the shareholders' ranking can be preserved only weakly in the limit (see Proposition 3. This means that under WAPP, at least one of the following must hold: (i) diffuse ownership is powerless in the sense that atomistic shareholders exert no control over the firm or (ii) when some of firm f's shareholders are atomistic, then every individual shareholder—whether large or atomistic—should have the same control power (i.e., zero). This suggests that under WAPP, there is a tension between (i) allowing for atomistic shareholders to (collectively) exert control over the firm and (ii) allowing for large shareholders to have control power.

With additional structure imposed on WAPP, the tension becomes more stark. Let $\gamma_{if}(s_{*f}) = \delta(s_{if}) / \sum_{j \in N} \delta(s_{jf})$ with $\delta(s_{if}) = s_{if}^{\alpha}$, $\alpha \geq 0$, as in Backus et al. (2021a) and Antón et al. (2023). $\alpha > 1$ is interpreted as large shareholders having disproportionately more power than smaller shareholders. $\alpha = 1$ corresponds to proportional control. $\alpha < 1$ is interpreted as large shareholders having less than proportionately more power than smaller shareholders. However, this interpretation of parameters of the firm's objective function can be at odds with the firm's conduct, since $\lambda_{fg}(s) \to \lambda_{i_1;fg}$ as $|N_2| \to \infty$ and $\max_{i \in N_2} s_{if} \to 0$ if $\alpha > 0$. If $\alpha = 0$, then $\lambda_{fg}(s)$ remains constant as ownership by N_2 is diffused. $\alpha = 0$ means that firm f maximizes the *unweighted* average of its shareholders' portfolio profits. This would imply firm f assigns weight

$$\lambda_{fg}(s) = \frac{\sum_{i \in N_f(\gamma_{*f})} \gamma_{if}(s_{*f}) s_{ig}}{\sum_{i \in N_f(\gamma_{*f})} \gamma_{if}(s_{*f}) s_{if}} = \frac{\sum_{i \in N_f(\gamma_{*f})} s_{ig} / |N_f(\gamma_{*f})|}{\sum_{i \in N_f(\gamma_{*f})} s_{if} / |N_f(\gamma_{*f})|} = \sum_{i \in N_f(\gamma_{*f})} s_{ig}$$

to firm g's profit, which can be unreasonably high. It is equal to 1 if firm g's shareholders are a subset of firm f's shareholders. If every firm's corporate control mechanism was such and every shareholder had some (however small or large) amount of shares of every firm in the industry, then each firm would assign weight 1 to the profit of every other firm, so that the firms acting as a multi-plant monopolist would be an equilibrium. To see why this can be particularly unrealistic, start from $s = I_n$, where I_n the identity matrix (*i.e.*, each firm is owned by a unique shareholder). If we slightly perturb s, so that each shareholder has some shares of every firm, the firms switch from own-profit maximization to collectively acting as a monopolist, a stark discontinuity.

Ownership diffusion under NB. Fix some s, α_{-f}, $i_1 \in N_1$, and $i_2 \in N_2$, and assume that for every α_f and $i \in N_1$, $j \in N_2$, $u_i(\alpha_f,\alpha_{-f},s_{i*}) - d_{if}(\alpha_{-f},s) = s_{if}/s_{i_1 f}(u_{i_1}(\alpha_f,\alpha_{-f},s_{i*}) -$

[10] For example, if A_f is compact and π_g is continuous for every firm g, then by Berge's Maximum Theorem, $R_f(\alpha_{-f},s) \equiv \arg\max_{\alpha_f} \left\{\pi_f(\alpha_f,\alpha_{-f}) + \sum_{g \in M \setminus \{f\}} \lambda_{fg}(s) \pi_g(\alpha_f,\alpha_{-f})\right\}$ is upper-hemicontinuous in λ_{f*}, so the limit of $R_f(\alpha_{-f},s)$ as $\lambda_{f*}(s) \to \lambda_{i_1;f*}$ (given that it exists) is a subset of $\arg\max_{\alpha_f} \left\{\pi_f(\alpha_f,\alpha_{-f}) + \sum_{g \in M \setminus \{f\}} \lambda_{i_1;fg} \pi_g(\alpha_f,\alpha_{-f})\right\}$.

$d_{i_1f}(\alpha_{-f},s))$, while for $i \in N_2$, $u_i(\alpha_f,\alpha_{-f},s_{i*}) - d_{if}(\alpha_{-f},s) = s_{if}/s_{i_2f}(u_{i_2}(\alpha_f,\alpha_{-f},s_{i*}) - d_{i_2f}(\alpha_{-f},s))$. This is, for example, satisfied if $d_{if}(\alpha_{-f},s) = u_i(\alpha_f^d(\alpha_{-f},s),\alpha_{-f},s_{i*})$, where $\alpha_f^d(\alpha_{-f},s)$ the strategy chosen by firm f in case of disagreement. Then, firm f's objective is

$$\prod_{i \in N_f(\beta_{*f}(s_{*f}))} \left(u_i\left(\alpha_f,\alpha_{-f},s_{i*}\right) - d_{if}(\alpha_{-f},s)\right)^{\beta_{if}(s_{*f})}$$
$$\propto \left(u_{i_1}(\alpha_f,\alpha_{-f},s_{i*}) - d_{i_1f}(\alpha_{-f},s)\right)^{\sum_{i\in N_1}\beta_{if}(s_{*f})} \left(u_{i_2}(\alpha_f,\alpha_{-f},s_{i*}) - d_{i_2f}(\alpha_{-f},s)\right)^{\sum_{i\in N_2}\beta_{if}(s_{*f})}.$$

Therefore, if we consider a sequence $s(\nu)_{\nu\in\mathbb{N}}$ of f-bianimous ownership matrices such that $|N_2| \to \infty$ and $\max_{i\in N_2} s_{if} \to 0$, the following two can hold at the same time: (i) firm f's strategy does *not* converge to the strategy most preferred by the group N_1 of shareholders and (ii) $\sum_{i\in N_1}\beta_{if}(s(\nu)_{*f})$ is bounded away from 0.[11] Thus, NB can relax the tension that arises under WAPP between (i) allowing for atomistic shareholders to (collectively) exert control over the firm and (ii) allowing for large shareholders to have control power.

4 Conclusion

Both theoretical and empirical work has so far followed the weighted average portfolio profit (WAPP) model put forward by Rotemberg (1984), Bresnahan and Salop (1986), and O'Brien and Salop (2000) to model corporate control under common ownership despite our limited understanding of the restriction it imposes on firm behavior. In this paper, I show that WAPP imposes two restrictions: (i) that the firm is efficiently controlled, and (ii) that the distribution of power across shareholders within the firm depends only on the firm's ownership structure, and not on external factors such as the stakes of the firm's shareholders in competing firms, the strategic choices of other firms, or market conditions (e.g., demand or production costs).

I propose the Nash bargaining (NB) model of corporate control under common ownership, a generalization of WAPP which models the firm's behavior as the result of asymmetric Nash bargaining among the firm's shareholders. NB also requires efficient control but allows for external factors to influence the distribution of power across the firm's shareholders. Indeed, I argue that external factors may play a role in the extent to which each shareholder controls the firm. In addition, I show that the NB model can relax the tension that arises under WAPP between (i) allowing for atomistic shareholders to collectively exert control over the firm while at the same time (ii) allowing for large shareholders to have control power. Last, I study the constraints imposed on the parameters of the firm's objective under WAPP and NB by additional restrictions on the

[11] For example, $R_f(\alpha_{-f},s(\nu))$ can be bounded away from the strategy most preferred by the group N_1 of shareholders if there exists $\kappa > 0$ such that $d_{i_2f}(\alpha_{-f},s(\nu)) > \max_{\alpha_f} u_{i_1}(\alpha_f,\alpha_{-f},s_{i*}) + \kappa$ and $\sum_{i\in N_2}\beta_{if}(s(\nu)_{*f}) > 0$ for every ν large enough.

firm's behavior, thereby characterizing a popular subclass of WAPP models.

This paper provides a unified axiomatic framework for studying corporate control under common ownership. It shows the general class of NB models of firm behavior is essentially characterized by an efficiency condition on firm behavior. WAPP models are the subclass of NB models that imposes an additional condition on firm behavior: the irrelevance of external factors for the distribution of control power across the firm's shareholders. The results guide researchers and practitioners to think in two steps when deciding on a model of corporate control under common ownership: (i) choose between WAPP and NB depending on whether the assumption that the distribution of control power across the firm's shareholders is independent of external factors is likely to be satisfied or not, and (ii) decide on a specific parametrization of the model chosen in the first step depending on what additional conditions are likely to be satisfied.

Future theoretical and empirical work can evaluate the robustness of results obtained under WAPP by also considering NB models of corporate control. For example, Backus et al. (2021b) find that own-firm profit maximization is more consistent with firm behavior in the ready-to-eat cereal market than WAPP with proportional control, which may serve as evidence against the "common ownership hypothesis" (i.e., that common ownership induces firms to internalize the effects of their strategic decisions on other firms' profits). An obvious robustness check would be to consider different parametrization of WAPP. However, if one is concerned that the distribution of power across shareholders may depend on external factors, this would not be enough. One would have to test NB models against own-profit maximization to more robustly evaluate the "common ownership hypothesis."

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Appendix

A Proofs

Proof of Proposition 1. Part (i). Let R_f be WAPP with control power function γ_{*f}. For every $s \in S$ define $\widetilde{N}(s_{*f}) = \{i \in N : \gamma_{if}(s_{*f}) > 0\}$, and use $\widetilde{N}(s_{*f})$ to verify that R_f satisfies the strong efficiency conditions.

Part (ii). Now, assume R_f is NB with bargaining power function β_{*f}. For every $s \in S$ define $\widetilde{N}(s_{*f}) \coloneqq \{i \in N : \beta_{if}(s_{*f}) > 0\}$, and use $\widetilde{N}(s_{*f})$ to verify that R_j satisfies the weak efficiency conditions.

Part (iii). Let R_f be NB with strict benefits form agreement and bargaining power function β_{*f}. For every $s \in S$ define $\widetilde{N}(s_{*f}) \coloneqq \{i \in N : \beta_{if}(s_{*f}) > 0\}$, and use $\widetilde{N}(s_{*f})$ to verify that R_f satisfies the strong efficiency conditions. Also, since there exists $u \in \mathcal{U}_f(\alpha_{-f}, s)$ such that $u_i > d_{if}$ for every $i \in N_f(\beta_{*f})$, the Nash product $\prod_{i \in N_f(\beta_{*f})} (u_i - d_{if})^{\beta_{if}}$ is strictly quasiconcave in u where that inequality holds. Thus, since $\{u \in \mathcal{U}_f(\alpha_{-f}, s) : u_i > d_{if}$ for every $i \in N_f(\beta_{*f})\}$ is convex for every $s \in S$ and $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$, there exists at most one u that maximizes the Nash product, so R_f is internally consistent.

Part (iv). Let R_f be weakly efficient and internally consistent, so that there exists function $\widetilde{N}(s_{*f})$ satisfying the conditions of Definition 3. For every $(\alpha_{-f}, s) \in \times_{g \neq f} \Delta(A_{-g}) \times S$ let the bargaining power function be

$$\beta_{*f}(s_{*f}) \coloneqq \frac{1}{\left|\widetilde{N}(s_{*f})\right|} \left(\mathbb{I}\left(1 \in \widetilde{N}(s_{*f})\right) \quad \ldots \quad \mathbb{I}\left(n \in \widetilde{N}(s_{*f})\right) \right),$$

where $\mathbb{I}$ the indicator function, and the disagreement payoff function be $d_{*f}(\alpha_{-f}, s) \coloneqq u(\widetilde{\alpha}_f(\alpha_{-f}, s), \alpha_{-f}, s)$ for some function $\widetilde{\alpha}_f(\alpha_{-f}, s)$ that is a selection from $R_f(\alpha_{-f}, s)$ (i.e., $\widetilde{\alpha}_f(\alpha_{-f}, s) \in R_f(\alpha_{-f}, s)$). d_{*f} is well-defined since R_f is internally consistent. Notice that by the way β_{*f} is defined, $N_f(\beta_{*f}) = \widetilde{N}(s_{*f})$. Observe that any $\alpha_f \in R_f(\alpha_{-f}, s)$ achieves the maximum value of zero for the Nash product, so

$$R_f(\alpha_{-f}, s) \subseteq \underset{\alpha_f \in B_f(a_{-f}, s)}{\arg\max} \left\{ \prod_{i \in N_f(\beta_{*f})} \left(u_f(\alpha_f, \alpha_{-f}, s_{i*}) - d_{if}(\alpha_{-f}, s)\right)^{\beta_{if}(s_{*f})} \right\}.$$

Now, take an arbitrary

$$\alpha_f \in \underset{\alpha_f \in B_f(a_{-f}, s)}{\arg\max} \left\{ \prod_{i \in N_f(\beta_{*f})} \left(u_i(\alpha_f, \alpha_{-f}, s_{i*}) - d_{if}(\alpha_{-f}, s)\right)^{\beta_{if}(s_{*f})} \right\}.$$

We will show by contradiction that $\alpha_f \in R_f(\alpha_{-f}, s)$. Assume that $\alpha_f \notin R_f(\alpha_{-f}, s)$. Then, since R_f is internally consistent, there exists $j \in N_f(\beta_{*f})$ such that $u_j(\alpha_f, \alpha_{-f}, s_{j*}) \neq$

$u_j\left(\widetilde{\alpha}_f(\alpha_{-f},s),\alpha_{-f},s_{j*}\right) = d_{jf}(\alpha_{-f},s)$. Also, given that α_f maximizes the Nash product above (and particularly, $\alpha_f \in B_f(a_{-f},s)$), $u_i\left(\alpha_f,\alpha_{-f},s_{f*}\right) \geq d_{if}(\alpha_{-f},s)$ for every $i \in N_f(\beta_{*f})$. Particularly, the inequality must hold strictly for j, that is, $u_j\left(\alpha_f,\alpha_{-f},s_{j*}\right) > u_j\left(\widetilde{\alpha}_f(\alpha_{-f},s),\alpha_{-f},s_{j*}\right) = d_{jf}(\alpha_{-f},s)$. But then, α_f weakly dominates $\widetilde{\alpha}_f(\alpha_{-f},s) \in R_f\left(\alpha_{-f},s\right)$, a contradiction to the weak efficiency of R_f. Therefore,

$$R_f\left(\alpha_{-f},s\right) \supseteq \underset{\alpha_f \in B_f(a_{-f},s)}{\arg\max} \left\{ \prod_{i \in N_f(\beta_{*f})} \left(u_f\left(\alpha_f,\alpha_{-f},s_{i*}\right) - d_{if}(\alpha_{-f},s)\right)^{\beta_{if}(s_{*f})} \right\}.$$

Part (v). Let R_f be WAPP with control power function γ_{*f}. For every $s \in S$ define $\widetilde{N}(s_{*f}) = \{i \in N : \gamma_{if}(s_{*f}) > 0\}$. Part (v) of Definition 3 is clearly satisfied. To see that part (iv) also holds, assume by contradiction that there exist $s \in S$, $\alpha_{-f} \in \times_{h \neq f} \Delta(A_{-h})$ and $\alpha_f,\alpha'_f \in R_f(\alpha_{-f},s)$, such that $u_j\left(\alpha_f,\alpha_{-f},s\right) \neq u_j\left(\alpha'_f,\alpha_{-f},s\right)$ for some shareholder j of firm f. Then, by strict convexity of $\mathcal{U}_f\left(\alpha_{-f}, s\right)$, any strict convex combination v of the two portfolio profit profiles under α_f and α'_f lies in the interior of $\mathcal{U}_f\left(\alpha_{-f}, s\right)$, and thus there exists $v' \in \mathcal{U}_f\left(\alpha_{-f}, s\right)$ such that $v' \gg v$, or equivalently α^*_f such that $u_i\left(\alpha^*_f,\alpha_{-f}, s\right) > v_i$ for every shareholder i of firm f. But then $\sum_{i \in N} \gamma_{if}(s_{*f}) u_i\left(\alpha^*_f, \alpha_{-f}, s_{i*}\right)$ is higher than the strict convex combination of $\sum_{i \in N} \gamma_{if}(s_{*f}) u_i\left(\alpha_f, \alpha_{-f}, s_{i*}\right)$ and $\sum_{i \in N} \gamma_{if}(s_{*f}) u_i\left(\alpha'_f, \alpha_{-f}, s_{i*}\right)$, and thus higher than each of the two (since $\alpha_f,\alpha'_f \in R_f(\alpha_{-f},s)$, so they both maximize the WAPP objective), which contradicts that $\alpha_f,\alpha'_f \in R_f(\alpha_{-f},s)$.

The proof for the case where R_f is NB is analogous. **Q.E.D.**

Proof of Proposition 2. Part (i). Since R_f is GWAPP, as in the proof of part (i) of Proposition 1, it is easy to see that R_f is strongly efficient. Part (iv) of Proposition 1 then implies that R_f is NB.

Part (ii). Assume that R_f is GWAPP. Then, as in the proof of parts (i) and (v) of Proposition 1, it is easy to see that R_f is strongly efficient and internally consistent. Part (iv) of Proposition 1 then implies that R_f is NB.

Now, assume that R_f is NB. Part (v) of Proposition 1 implies that it is strongly efficient and internally consistent. Since R_f is strongly efficient, by the separating hyperplane theorem, for every (α_{-f},s), there exists non-zero $\gamma_{*f}(\alpha_{-f},s) \in \mathbb{R}^{|\widetilde{N}(s_{*f})|}$, where $\widetilde{N}(s_{*f})$ the nonempty set of controlling shareholders of Definition 3, such that for every $\alpha_f \in R_f(\alpha_{-f},s)$

$$\max_{v \in \mathcal{U}_f\left(\alpha_{-f},s\right)} \left\{ \sum_{i \in \widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s) v_i \right\} = \sum_{i \in \widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s) u_i(\alpha_f, \alpha_{-f}, s_{i*}) \tag{1}$$

and $\sum_{i \in \widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s) v_i \geq \max_{u \in \mathcal{U}_f\left(\alpha_{-f},s\right)} \left\{ \sum_{i \in \widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s) v_i \right\}$ for every v such that $v_i > u_i(\alpha_f, \alpha_{-f}, s_{i*})$ for every $i \in \widetilde{N}(s_{*f})$.[12] Particularly, it must be that $\gamma_{*f}(\alpha_{-f},s) \in$

[12]We have also used the fact that $u_i(\alpha_f, \alpha_{-f}, s_{i*}) = u_i(\alpha'_f, \alpha_{-f}, s_{i*})$ for every $i \in \widetilde{N}(s_{*f})$ and every

$\mathbb{R}_+^{|\widetilde{N}(s_{*f})|}$ for if $\gamma_{jf}(\alpha_{-f},s) < 0$ for some controlling shareholder j, then $\sum_{i\in\widetilde{N}(s_{*f})}\gamma_{if}(\alpha_{-f},s)v_i \geq \max_{u\in\mathcal{U}_f(\alpha_{-f},s)}\left\{\sum_{i\in\widetilde{N}(s_{*f})}\gamma_{if}(\alpha_{-f},s)v_i\right\}$ will be violated for v such that v_j is large enough. Also, $\gamma_{*f}(\alpha_{-f},s) \in \mathbb{R}_+^{|\widetilde{N}(s_{*f})|}$ can be normalized so that its entries sum up to 1. Therefore,

$$R_f\left(\alpha_{-f},s\right) \subseteq \underset{\alpha_f\in\Delta(A_f)}{\arg\max}\left\{\sum_{i\in\widetilde{N}(s_{*f})}\gamma_{if}(\alpha_{-f},s)u_i(\alpha_f,\alpha_{-f},s_{i*})\right\}.$$

Also, because $\mathcal{U}_f\left(\alpha_{-f},s\right)$ is strictly convex, for any

$$\alpha_f,\alpha_f' \in \underset{\alpha_f\in\Delta(A_f)}{\arg\max}\left\{\sum_{i\in\widetilde{N}(s_{*f})}\gamma_{if}(\alpha_{-f},s)u_i(\alpha_f,\alpha_{-f},s_{i*})\right\},$$

it holds that $u_i(\alpha_f,\alpha_{-f},s_{i*}) = u_i(\alpha_f',\alpha_{-f},s_{i*})$ for every $i\in\widetilde{N}(s_{*f})$, and particularly, given also (1), $u_i(\alpha_f,\alpha_{-f},s_{i*}) = u_i(\alpha_f'',\alpha_{-f},s_{i*})$ for every $i\in\widetilde{N}(s_{*f})$ and every $\alpha_f''\in R_f\left(\alpha_{-f},s\right)$. Therefore, given that R_f is internally consistent (part (v) of Definition 3)

$$R_f\left(\alpha_{-f},s\right) \supseteq \underset{\alpha_f\in B_f(a_{-f},s)}{\arg\max}\left\{\prod_{i\in N_f(\beta_{*f})}\left(u_f\left(\alpha_f,\alpha_{-f},s_{i*}\right) - d_{if}(\alpha_{-f},s)\right)^{\beta_{if}(s_{*f})}\right\}.$$

Q.E.D.

Proof of Proposition 3. (i) Firm f's objective function is $\sum_{k\in N_f(\gamma_{*f}(s'_{*f}))}\gamma_{kf}(s'_{*f})u_k(a_f,a_{-f},s'_{k*})$, and the Implicit Function Theorem gives

$$\nabla_{ds}R_f(a_{-f},s) = -\frac{\left(\gamma_{if}(s_{*f}) - \gamma_{jf}(s_{*f})\right)\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}}{\sum_{k\in N_j(\beta_{*j})}\gamma_{kf}(s_{*f})\left.\frac{\partial^2u_k(a_f,a_{-f},s_{k*})}{\partial a_f^2}\right|_{a_f=R_f(a_{-f},s)}} \implies$$

$$\operatorname{sgn}\left\{\nabla_{ds}R_f(a_{-f},s)\right\} = \operatorname{sgn}\left\{\left(\gamma_{if}(s_{*f}) - \gamma_{jf}(s_{*f})\right)\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}\right\}. \quad (2)$$

We will prove each direction separately.

$\Longleftarrow$: Let R_f be WAPP and assume that for every s and every pair of firm f shareholders $i,j\in N_f(s_{*f})$, $s_{if}\geq s_{jf} \implies \gamma_{if}(s_{*f})\geq\gamma_{jf}(s_{*f})$. We need to show that R_f is rank-preserving. Take arbitrary $a_{-f}\in A_{-f}$, f-unanimous $s\in S$, and pair of distinct shareholders $i,j\in N_f(s_{*f})$ with $s_{if}>s_{jf}$. Consider a stock trade where i buys firm $g\neq f$ shares from j. (2) combined with the fact that $s_{if}>s_{jf}\implies\gamma_{if}(s_{*f})\geq\gamma_{jf}(s_{*f})$ gives

$$\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)} \overset{(\text{resp. }\leq)}{\geq} 0 \implies \nabla_{ds}R_f(a_{-f},s) \overset{(\text{resp. }\leq)}{\geq} 0.$$

$\alpha_f,\alpha_f'\in R_f(\alpha_{-f},s)$ since R_f is internally consistent (part (iv) of Definition 3).

$\Longrightarrow$: Let R_f be WAPP and rank-preserving. We need to show that for every s and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$. Take arbitrary $s \in S$ and pair of firm f shareholders $i,j \in N_f(s_{*f})$ with $s_{if} \geq s_{jf}$. Clearly, there exists f-unanimous $\widehat{s} \in S$ such that $\widehat{s}_{*f} = s_{*f}$. Also, by assumption, there exist firm $g \neq f$ and $\widehat{a}_{-f} \in A_{-f}$ such that $\partial\pi_g(a_f,\widehat{a}_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} \neq 0$. Let $ds := (\mathbf{e}_i - \mathbf{e}_j) \otimes \mathbf{e}_g$. If $\partial\pi_g(a_f,a_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} > 0$, then given that R_f is rank-preserving, $\nabla_{ds}R_f(a_{-f},s) \geq 0$. Given $s_{if} \geq s_{jf}$, (2) implies that $\gamma_{if}(\widehat{s}_{*f}) \geq \gamma_{jf}(\widehat{s}_{*f})$, which given that $\widehat{s}_{*f} = s_{*f}$, implies that $\gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$. Similarly, if $\partial\pi_g(a_f,a_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} < 0$, then given that R_f is rank-preserving, $\nabla_{ds}R_f(a_{-f},s) \leq 0$. Given $s_{if} \geq s_{jf}$, (2) implies that $\gamma_{if}(\widehat{s}_{*f}) \geq \gamma_{jf}(\widehat{s}_{*f})$, which given that $\widehat{s}_{*f} = s_{*f}$, implies that $\gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$.

(ii) Now, notice that under NB, for any f-unanimous s, $\partial u_k(a_f,a_{-f},s_{k*})/\partial a_f|_{a_f=R_f(a_{-f},s)} = 0$ for every shareholder k of firm f. For a stock trade ds, the Implicit Function Theorem then gives[13]

$$\begin{aligned}
\operatorname{sgn}\{\nabla_{ds}R_f(a_{-f},s)\} &= \operatorname{sgn}\left\{ (\widetilde{\gamma}_{if}(a_{-f},s) - \widetilde{\gamma}_{jf}(a_{-f},s)) \left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)} \right\} \\
&= \operatorname{sgn}\left\{ \frac{(\beta_{if}(s_{*f}) - \beta_{jf}(s_{*f})s_{if}/s_{jf}) \left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}}{u_i\left(R_f(a_{-f},s),a_{-f},s_{i*}\right) - u_i\left(\alpha_d(a_{-f},s),a_{-f},s_{i*}\right)} \right\} \\
&= \operatorname{sgn}\left\{ \left(\frac{\beta_{if}(s_{*f})}{s_{if}} - \frac{\beta_{jf}(s_{*f})}{s_{jf}}\right) \left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)} \right\}, \qquad (3)
\end{aligned}$$

where $\alpha_d(a_{-f},s)$ the strategy followed in case of disagreement. In the second line, we have used the fact that $\lambda_{i;f*} = \lambda_{j;f*}$, and in the third the fact that $u_i\left(R_f(a_{-f},s),a_{-f},s_{i*}\right) > u_i\left(\alpha_d(a_{-f},s),a_{-f},s_{i*}\right)$. We will prove each direction separately.

$\Longleftarrow$: Let R_f be NB and assume that for every s and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \beta_{if}(s_{*f})/s_{if} \geq \beta_{jf}(s_{*f})/s_{jf}$. We need to show that R_f is rank-preserving. Take arbitrary $a_{-f} \in A_{-f}$, f-unanimous $s \in S$, and pair of distinct shareholders $i,j \in N_f(s_{*f})$ with $s_{if} \geq s_{jf}$. Consider a stock trade where i buys firm $g \neq f$ shares from j. (3) combined with the fact that $s_{if} \geq s_{jf} \implies \beta_{if}(s_{*f})/s_{if} \geq \beta_{jf}(s_{*f})/s_{jf}$ gives

$$\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)} \overset{\text{(resp. }\leq)}{\geq} 0 \implies \nabla_{ds}R_f(a_{-f},s) \overset{\text{(resp. }\leq)}{\geq} 0.$$

$\Longrightarrow$: Let R_f be NB and rank-preserving. We need to show that for every s and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \beta_{if}(s_{*f})/s_{if} \geq \beta_{jf}(s_{*f})/s_{jf}$. Take arbitrary $s \in S$ and pair of firm f shareholders $i,j \in N_f(s_{*f})$ with $s_{if} \geq s_{jf}$. Clearly, there

[13] Notice that because $\partial u_k(a_f,a_{-f},s_{k*})/\partial a_f|_{a_f=R_f(a_{-f},s)} = 0$ for every shareholder k of firm f, the changes in $\widetilde{\gamma}_{*f}$ caused by the stock trade vanish.

exists f-unanimous $\widehat{s} \in S$ such that $\widehat{s}_{*f} = s_{*f}$. Also, by assumption, there exist firm $g \neq f$ and $\widehat{a}_{-f} \in A_{-f}$ such that $\partial\pi_g(a_f,\widehat{a}_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} \neq 0$. Let $ds := (\mathbf{e}_i - \mathbf{e}_j) \otimes \mathbf{e}_g$. If $\partial\pi_g(a_f,a_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} > 0$, then given that R_f is rank-preserving, $\nabla_{ds}R_f(a_{-f},s) \geq 0$. Given $s_{if} \geq s_{jf}$, (3) then implies that $\beta_{if}(s_{*f})/s_{if} \geq \beta_{jf}(s_{*f})/s_{jf}$, which given that $\widehat{s}_{*f} = s_{*f}$, implies that $\beta_{if}(\widehat{s}_{*f})/\widehat{s}_{if} \geq \beta_{jf}(\widehat{s}_{*f})/\widehat{s}_{jf}$. Similarly, if $\partial\pi_g(a_f,a_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} < 0$, then given that R_f is rank-preserving, $\nabla_{ds}R_f(a_{-f},s) \leq 0$. Given $s_{if} \geq s_{jf}$, (3) then implies that $\beta_{if}(\widehat{s}_{*f})/\widehat{s}_{if} \geq \beta_{jf}(\widehat{s}_{*f})/\widehat{s}_{jf}$, which given that $\widehat{s}_{*f} = s_{*f}$, implies that $\beta_{if}(\widehat{s}_{*f})/\widehat{s}_{if} \geq \beta_{jf}(\widehat{s}_{*f})/\widehat{s}_{jf}$. **Q.E.D.**

Proof of Lemma 1. (i-a) Firm f's objective function is $\sum_{k\in N_f(\gamma_{*f}(s'_{*f}))} \gamma_{kf}(s_{*f})u_k(a_f,a_{-f},s_{k*})$, and the Implicit Function Theorem gives

$$\operatorname{sgn}\{\nabla_{ds}R_f(a_{-f},s)\} = \operatorname{sgn}\left\{\left((1-\psi)\gamma_{if}(s_{*f}) - \psi\sum_{j\in N_f(s_{*f})\setminus\{i\}}\gamma_{jf}(s_{*f})\right)\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}\right\}. \tag{4}$$

We will prove each direction separately.

$\Longleftarrow$: Let R_f be WAPP, and assume that $\gamma_{if}(s_{*f}) = \psi$. We need to show that the stock trade is neutral. Take arbitrary $a_{-f} \in A_{-f}$. (4) then gives

$$\operatorname{sgn}\{\nabla_{ds}R_f(a_{-f},s)\} = \operatorname{sgn}\left\{((1-\psi)\psi - \psi(1-\psi)))\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}\right\} = 0.$$

$\Longrightarrow$: Let R_f be WAPP, and assume that the stock trade is neutral. We need to show that $\gamma_{if}(s_{*f}) = \psi$. By assumption, there exists and $\widehat{a}_{-f} \in A_{-f}$ such that $\partial\pi_g(a_f,\widehat{a}_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} \neq 0$. Then, (4) implies that $\gamma_{if}(s_{*f}) = \psi/(1+\psi)$.

(ii-a) Now, notice that under NB, for any f-unanimous s, the Implicit Function Theorem then gives

$$\begin{aligned}\operatorname{sgn}\{\nabla_{ds}R_f(a_{-f},s)\} &= \operatorname{sgn}\left\{\left((1-\psi)\widetilde{\gamma}_{if}(a_{-f},s) - \psi\sum_{j\in N_f(s_{*f})\setminus\{i\}}\widetilde{\gamma}_{jf}(a_{-f},s)\right)\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}\right\}\\ &= \operatorname{sgn}\left\{\left((1-\psi)\frac{\beta_{if}(s_{*f})}{s_{if}} - \psi\sum_{j\in N_f(s_{*f})\setminus\{i\}}\frac{\beta_{jf}(s_{*f})}{s_{jf}}\right)\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}\right\},\end{aligned}$$

where the second line follows as in the proof of Proposition 3. Then, the result follows as in part (i).

Similar arguments prove (i-b) and (ii-b). **Q.E.D.**

Proof of Proposition 4. (i) $\Rightarrow$: Assume that R_f has monotone control power, and take arbitrary s, pair of firm f shareholders $i,j \in N_f(s_{*f})$, and s' such that $s'_{*f} = s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)$

for some $t \in [0, \min\{s_{jf}, 1 - s_{if}\}]$. Clearly, there exist f-unanimous ownership matrices $\widehat{s},\widehat{s}' \in S$ such that $\widehat{s}_{*f} = s_{*f}$, $\widehat{s}'_{*f} = s'_{*f}$, $\widehat{s}'_{k*} = \widehat{s}_{k*}$ for every $k \neq i,j$, $\widehat{s}'_{i*} = (1+t/s_{if})\widehat{s}_{i*}$, and $\widehat{s}'_{j*} = \widehat{s}_{j*} - t/s_{if}\widehat{s}_{i*}$. Given Lemma 1, starting from $\widehat{s}$, a $(\gamma_{if}(\widehat{s}_{*f}),g,i,N_f(\widehat{s}_{*f}) \setminus \{i\})$-stock trade is f-neutral. Also, starting from $\widehat{s}'$, a $(\gamma_{if}(\widehat{s}'_{*f}),g,i,N_f(\widehat{s}'_{*f}) \setminus \{i\})$-stock trade is f-neutral. Thus, given that R_f has monotone control power, $\gamma_{if}(\widehat{s}'_{*f}) \geq \gamma_{if}(\widehat{s}_{*f})$, which given that $\widehat{s}_{*f} = s_{*f}$ and $\widehat{s}'_{*f} = s'_{*f}$, implies $\gamma_{if}(s'_{*f}) \geq \gamma_{if}(s_{*f})$.

$\Leftarrow$: Assume that for every s, every pair of firm f shareholders $i,j \in N_f(s_{*f})$, and $t \in [0, \min\{s_{jf}, 1 - s_{if}\}]$, $\gamma_{if}(s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)) \geq \gamma_{if}(s_{*f})$. Now, take arbitrary $g \neq f$, pair of shareholders $i,j \in N$, and f-unanimous ownership matrix s. Assume that starting from s, a $(\psi,g,i,N_f(s_{*f}) \setminus \{i\})$-stock trade is f-neutral, and that starting from s' such that $s'_{k*} = s_{k*}$ for every $k \neq i,j$, $s'_{i*} = (1 + t)s_{i*}$, and $s'_{j*} = s_{j*} - ts_{i*}$ for some $t \in [0, \min_{g \in M: s_{ig} > 0} \min\{1 - s_{ig}/s_{ig}, s_{jg}/s_{ig}\}]$, a $(\psi',g,i,N_f(s'_{*f}) \setminus \{i\})$-stock trade is f-neutral. Then, given Lemma 1, $\psi = \gamma_{if}(s_{*f})$ and $\psi' = \gamma_{if}(s'_{*f})$. Therefore, $\psi' \geq \psi$.

Part (ii) follows similarly. **Q.E.D.**

Proof of Proposition 5. (i) $\Leftarrow$: Assume that there exists non-decreasing $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\delta(0) = 0$ and $\delta(x) > 0$ for every $x > 0$ such that for every s and every $i \in N$, $\gamma_{if}(s_{*f}) = \delta(s_{if}) / \sum_{j \in N} \delta(s_{jf})$ holds. Clearly, R_f is anonymous. To see why it is inclusive, take arbitrary firm $g \neq f$, f-unanimous ownership matrix s, and pair of firm f's shareholders $i,j \in N_f(s_{*f})$.[14] Lemma 1 then implies that a $(0,g,i,j)$-stock trade is not f-neutral since $\gamma_{if}(s_{*f})(1 - 0) = \delta(s_{if}) / \sum_{k \in N} \delta(s_{kf}) > 0 = \gamma_{jf}(s_{*f}) \times 0$,

It remains to show that R_f satisfies IIS. Take arbitrary $g \neq f$, f-unanimous ownership matrices s and s', and pair of shareholders $i,j \in N_f(s_{*f})$ such that $s'_{if} = s_{if}$ and $s'_{jf} = s_{jf}$, and $\psi \in [0,1]$. Take any (ψ,g,i,j)-stock trade that is f-neutral starting from s. Lemma 1 implies $\gamma_{if}(s_{*f})(1 - \psi) = \gamma_{jf}(s_{*f})\psi$. Multiplying both sides by $\gamma_{jf}(s'_{*f})$, we get $\gamma_{if}(s_{*f})\gamma_{jf}(s'_{*f})(1 - \psi) = \gamma_{jf}(s_{*f})\gamma_{jf}(s'_{*f})\psi$, which given that $\gamma_{if}(s_{*f})\gamma_{jf}(s'_{*f}) = \gamma_{if}(s'_{*f})\gamma_{jf}(s_{*f})$, implies $\gamma_{if}(s'_{*f})\gamma_{jf}(s_{*f})(1-\psi) = \gamma_{jf}(s_{*f})\gamma_{jf}(s'_{*f})\psi$. Given that $\gamma_{jf}(s_{*f}) = \delta(s_{jf}) / \sum_{k \in N} \delta(s_{kf}) > 0$, we can divide both sides by $\gamma_{jf}(s_{*f})$, which gives $\gamma_{if}(s'_{*f})(1-\psi) = \gamma_{jf}(s'_{*f})\psi$. Therefore, given Lemma 1, the (ψ,g,i,j)-stock trade is f-neutral also starting from s'.

$\Rightarrow$: Assume that R_f satisfies anonymity, inclusivity, stock-trade monotonicity, and IIS. That R_f is WAPP implies that there exists $\gamma_{*f} : \Delta^n \to \Delta^n$ satisfying the conditions of definition 1. By anonymity, for every α_{-f}, s, and permutation matrix P,

$$\arg\max_{a_f \in A_f} \left\{ \sum_{k \in N} \gamma_{kf}(s_{*f}) u_k(a_f, \alpha_{-f}, s_{k*}) \right\} = \arg\max_{a_f \in A_f} \left\{ \sum_{k \in N} \gamma_{kf}(Ps_{*f}) u_k(a_f, \alpha_{-f}, (Ps)_{k*}) \right\}. \tag{5}$$

[14]If f has only one shareholder, then R_f is automatically inclusive.

Also, pre-multiplying $\gamma_{*f}(s_{*f})$ and s by P simply relabels firm f's shareholders, so that for every α, s, and permutation matrix P,

$$\sum_{k\in N}\gamma_{kf}(s_{*f})u_k(a_f,\alpha_{-f},s_{k*}) = \sum_{k\in N}(P\gamma_{*f}(s_{*f}))_k u_k(a_f,\alpha_{-f},(Ps)_{k*}).$$

This combined with (5) implies that without loss, we can let $\gamma_{*f}:\Delta^n\to\Delta^n$ be such that for every s and P, $\gamma_{*f}(Ps_{*f}) = P\gamma_{*f}(s_{*f})$.[15]

Now, notice that for any s and $i\in N_f(s_{*f})$, $\gamma_{if}(s_{*f})>0$. To see this, take arbitrary s and $i\in N_f(s_{*f})$. Clearly, there exists f-unanimous $\widehat{s}\in S$ such that $\widehat{s}_{*f}=s_{*f}$. By inclusivity, for any firm $g\neq f$ and shareholder $j\in N_f(s_{*f})\setminus\{i\}$,[16] a $(0,g,i,j)$-stock trade is not f-neutral, which by Lemma 1 means that $\gamma_{if}(\widehat{s}_{*f})\neq 0$. Given that $\widehat{s}_{*f}=s_{*f}$, this implies that $\gamma_{if}(s_{*f})>0$.

Now, take arbitrary s, s', and pair of shareholders $i,j\in N_f(s_{*f})$ such that $s'_{if}=s_{if}$ and $s'_{jf}=s_{jf}$. Clearly, there exists f-unanimous $\widehat{s}\in S$ such that $\widehat{s}_{*f}=s_{*f}$ and f-unanimous $\widehat{s}'\in S$ such that $\widehat{s}'_{*f}=s'_{*f}$. Since R_f satisfies IIS, for any $\psi\in[0,1]$, if starting from $\widehat{s}$, a (ψ,g,i,j)-stock trade is f-neutral, then starting from $\widehat{s}'$, a (ψ,g,i,j)-stock trade is f-neutral. Given Lemma 1, starting from $\widehat{s}$, a (ψ,g,i,j)-stock trade is f-neutral if and only if $(1-\psi)\gamma_{if}(\widehat{s}_{*f})=\psi\gamma_{jf}(\widehat{s}_{*f})$. Similarly, starting from $\widehat{s}'$, a (ψ,g,i,j)-stock trade is f-neutral if and only if $(1-\psi)\gamma_{if}(\widehat{s}'_{*f})=\psi\gamma_{jf}(\widehat{s}'_{*f})$. Therefore, we have that for every $\psi\in[0,1]$, if $(1-\psi)\gamma_{if}(\widehat{s}_{*f})=\psi\gamma_{jf}(\widehat{s}_{*f})$, then $(1-\psi)\gamma_{if}(\widehat{s}'_{*f})=\psi\gamma_{jf}(\widehat{s}'_{*f})$. It holds that $(1-\psi)\gamma_{if}(\widehat{s}_{*f})=\psi\gamma_{jf}(\widehat{s}_{*f})$ if and only if $\psi=\psi^*:=\gamma_{if}(\widehat{s}_{*f})/(\gamma_{if}(\widehat{s}_{*f})+\gamma_{jf}(\widehat{s}_{*f}))$. Substituting ψ^* in $(1-\psi)\gamma_{if}(\widehat{s}'_{*f})=\psi\gamma_{jf}(\widehat{s}'_{*f})$, we get $\gamma_{if}(\widehat{s}_{*f})\gamma_{jf}(\widehat{s}'_{*f})=\gamma_{if}(\widehat{s}'_{*f})\gamma_{jf}(\widehat{s}_{*f})$, which given that $\widehat{s}_{*f}=s_{*f}$ and $\widehat{s}'_{*f}=s'_{*f}$, implies $\gamma_{if}(s_{*f})/\gamma_{jf}(s_{*f})=\gamma_{if}(s'_{*f})/\gamma_{jf}(s'_{*f})$. Therefore, there exists function $h_{ij}:\{(x,y)\in\mathbb{R}^2_{++}:x+y\leq 1\}\to\mathbb{R}_{++}$ such that $\gamma_{if}(s_{*f})=h_{ij}(s_{if},s_{jf})\gamma_{jf}(s_{*f})$ for every s. Given that for every s and P, $\gamma_{*f}(Ps_{*f})=P\gamma_{*f}(s_{*f})$, we can drop the subscript ij from h; namely, there exists $h:\{(x,y)\in\mathbb{R}^2_{++}:x+y\leq 1\}\to\mathbb{R}_{++}$ such that $h(s_{if},s_{jf})=\gamma_{if}(s_{*f})/\gamma_{jf}(s_{*f})$ for every s and every $i,j\in N_f(s_{*f})$. Notice that for every s and every $i,j,k\in N_f(s_{*f})$

$$h(s_{if},s_{jf})=\frac{\gamma_{if}(s_{*f})}{\gamma_{jf}(s_{*f})}=\frac{\gamma_{if}(s_{*f})/\gamma_{kf}(s_{*f})}{\gamma_{jf}(s_{*f})/\gamma_{kf}(s_{*f})}=\frac{h(s_{if},s_{kf})}{h(s_{jf},s_{kf})}.$$

This means that for every $x,y,z>0$ such that $x+y+z\leq 1$, $h(x,y)=h(x,z)/h(y,z)$. In fact, this equation must hold more generally. To see this, take arbitrary $x,y,z>0$ such

[15] In more detail, it implies that, even if the function $\gamma_{*f}:\Delta^n\to\Delta^n$ satisfying the conditions of definition 1 is not unique, there exists $\gamma'_{*f}:\Delta^n\to\Delta^n$ that satisfies the conditions of definition 1 and at the same time is such that for every s and P, $\gamma'_{*f}(Ps_{*f})=P\gamma'_{*f}(s_{*f})$.

[16] If $s_{if}=1$, then automatically $\gamma_{if}(\widehat{s}_{*f})=1$.

that $x+y<1$, $x+z<1$ and $y+z<1$. It then holds that

$$\begin{aligned}
h(x,y) &= \frac{h(x,1-\max\{x+y,x+z,y+z\})}{h(y,1-\max\{x+y,x+z,y+z\})} \\
&= \frac{h(x,1-\max\{x+y,x+z,y+z\})/h(z,1-\max\{x+y,x+z,y+z\})}{h(y,1-\max\{x+y,x+z,y+z\})/h(z,1-\max\{x+y,x+z,y+z\})} \\
&= \frac{h(x,z)}{h(y,z)}, \qquad (6)
\end{aligned}$$

where the first and third lines follow from $h(x,y) = h(x,z)/h(y,z)$ holding for every $x,y,z>0$ such that $x+y+z \leq 1$.

Now, define $\delta : [0,1] \to \mathbb{R}_+$ given by

$$\delta(x) \coloneqq \begin{cases} 0 & \text{if } x = 0 \\ h(x, 1/5) & \text{if } x \in (0, 3/4] \\ \frac{h(x,(1-x)/5)}{h(1/5,(1-x)/5)} & \text{if } x \in (3/4, 1) \end{cases}$$

and satisfying $\delta(x)/\delta(y) = h(x,y)$ for all $x, y \in (0,1)$ such that $x+y<1$. To see this, notice that:

1. If $x, y \in (0, 3/4]$, then from (6) it follows that

$$\frac{\delta(x)}{\delta(y)} = \frac{h(x, 1/5)}{h(y, 1/5)} = h(x, y).$$

2. If $x \in (3/4, 1)$ (and thus $y \in (0, 1/4)$), then

$$\frac{\delta(x)}{\delta(y)} = \frac{\frac{h(x,(1-x)/5)}{h(1/5,(1-x)/5)}}{h(y, 1/5)} = \frac{h(x,(1-x)/5)}{h(1/5,(1-x)/5)h(y, 1/5)},$$

   where $h(1/5,(1-x)/5) = 1/h((1-x)/5,1/5)$ and, given (6), $h(x,(1-x)/5) = h(x,y)/h((1-x)/5,y)$, so

$$\frac{\delta(x)}{\delta(y)} = \frac{h(x,y)}{h((1-x)/5,y)} \frac{h((1-x)/5,1/5)}{h(y, 1/5)} = \frac{h(x,y)}{h((1-x)/5,y)} h((1-x)/5,y) = h(x, y),$$

   where the second equality also follows from (6).

3. If $y \in (3/4, 1)$ (and thus $x \in (0, 1/4)$), then $\delta(x)/\delta(y) = (\delta(y)/\delta(x))^{-1} = (h(y, x))^{-1} = h(x,y)$, where the second equality also follows from the previous case.

We have then that for every s and distinct $i,j \in N_f(s_{*f})$ such that $s_{if} + s_{jf} < 1$, $\gamma_{if}(s_{*f})/\gamma_{jf}(s_{*f}) = \delta(s_{if})/\delta(s_{jf})$. This equality also automatically holds when $j \in N_f(s_{*f})$

but $i \notin N_f(s_{*f})$. Therefore, for every s such that $|N_f(s_{*f})| \geq 3$ and every $j \in N_f(s_{*f})$

$$1 = \sum_{i \in N} \gamma_{if}(s_{*f}) = \sum_{i \in N} \frac{\delta(s_{if})}{\delta(s_{jf})} \gamma_{jf}(s_{*f}) = \frac{\gamma_{jf}(s_{*f})}{\delta(s_{jf})} \sum_{i \in N} \delta(s_{if}) \implies \gamma_{jf}(s_{*f}) = \frac{\delta(s_{jf})}{\sum_{i \in N} \delta(s_{if})}.$$

Also, for $j \notin N_f(s_{*f})$, it automatically holds that $\gamma_{jf}(s_{*f}) = \delta(s_{jf}) / \sum_{i \in N} \delta(s_{if}) = \delta(0) / \sum_{i \in N} \delta(s_{if}) = 0$. **Q.E.D.**

Online Appendix

Corporate control under common ownership

Orestis Vravosinos

B The weighted average profit weight (WAPW) mechanism

In Brito et al.'s (2023) voting model, when the profit relevance of shareholder bias parameter is equal to 1, the authors frame the corporate control mechanism as—what I call—a weighted average profit weight (WAPW) mechanism.

Definition 15. Firm f's corporate control mechanism R_f is a weighted average profit weight (WAPW) if there exists a control power function $\widehat{\gamma}_{*f} : \Delta^n \to \Delta^n$ such that for every $s \in S$ and $\alpha_{-f} \in \times_{g \neq f} \Delta(A_{-g})$

(i) (*weighted sum of firm profit maximization with weighted average profit weights*)

$$R_f\left(\alpha_{-f}, s\right) = \underset{\alpha_f \in \Delta(A_f)}{\arg\max} \left\{ \pi_f(\alpha_f, \alpha_{-f}) + \sum_{g \in M \setminus \{f\}} \left(\sum_{i \in N_f(\gamma_{*f})} \widehat{\gamma}_{if}(s_{*j}) \lambda_{i;fg} \right) \pi_g(\alpha_f, \alpha_{-f}) \right\},$$

where $N_f(\gamma_{*f}(s_{*f})) \equiv \{i \in N : \widehat{\gamma}_{if}(s_{*f}) > 0\}$,

(ii) (*control exclusive to shareholders*) for every $i \in N$, $s_{if} = 0 \implies \widehat{\gamma}_{if}(s_{*f}) = 0$.

In WAPW, the weight that the manager of firm f places on firm g's profit is a weighted average of the weights $\{\lambda_{i;fg}\}_{i \in N_f(s_{*f})}$ that the shareholders of firm f would want firm f to use. This still is a WAPP mechanism, since it can be written as

$$R_f\left(\alpha_{-f}, s\right) = \underset{\alpha_f \in \Delta(A_f)}{\arg\max} \left\{ \sum_{i \in N_f(\widehat{\gamma}_{*f})} \gamma_{if}(s_{*f}) u_i(\alpha_f, \alpha_{-f}, s_{i*}) \right\},$$

where for every shareholder i of firm f

$$\gamma_{if}(s_{*f}) := \frac{\widehat{\gamma}_{if}(s_{*f})/s_{if}}{\sum_{i \in N_f(\widehat{\gamma}_{*f})} \widehat{\gamma}_{if}(s_{*f})/s_{if}}.$$

Thus, a mechanism is WAPP if and only if it is WAPW.

The novelty is that the WAPW parametrizations considered in Brito et al. (2023) give rise to γ's that are not standard in the literature. If all shares have voting rights, proportional $\widehat{\gamma}$'s give rise to

$$\gamma_{if}(s_{*j}) = \begin{cases} 1/|N_f(s_{*f})| & \text{if } s_{if} > 0 \\ 0 & \text{if } s_{if} = 0, \end{cases}$$

while Banzhaf $\widehat{\gamma}$'s give rise to

$$\gamma_{if}(s_{*f}) := \begin{cases} \frac{\gamma^B_{if}(s_{*f})/s_{if}}{\sum_{j\in N_f(s_{*f})}\gamma^B_{jf}(s_{*f})/s_{jf}} & \text{if } s_{if}(s_{*f}) > 0 \\ 0 & \text{if } s_{if}(s_{*f}) = 0, \end{cases}$$

$$\gamma^B_{if}(s_{*f}) = \frac{\left|\left\{T \in 2^N : \sum_{k\in T} s_{kf} \geq 1/2 > \sum_{k\in T\setminus\{i\}} s_{kf}\right\}\right|}{\sum_{t\in N}\left|\left\{T \in 2^N : \sum_{k\in T} s_{kf} \geq 1/2 > \sum_{k\in T\setminus\{t\}} s_{kf}\right\}\right|}.$$

C The random dictatorship disagreement payoff function

The random dictatorship specification of the disagreement payoff function poses that in case of disagreement in a firm, the shareholders' payoffs are derived from random dictatorship: With some exogenous probability, each shareholder of the firm is chosen to implement her most preferred strategy.

Definition 16. The disagreement payoff function d_{*f} is a random dictatorship (RD) disagreement payoff function if there exist a lottery weight function $\delta_{*f} : \Delta^n \to \Delta^n$ and a choice function (in case of disagreement) $\alpha^d_f : \times_{g\neq f}\Delta(A_k) \times \{v \in \mathbb{R}^m_+ : v_f = 1\} \to \Delta(A_f)$ such that

(i) (*the choice function α^d_f for firm f is a selection from the correspondence that takes as arguments the other firms' strategies α_{-f} and a vector v of relative weights on firms' profits (with the weight on firm f's profit normalized to* 1*) and returns the firm f strategies that maximize the payoff of a shareholder with relative holdings v in the firms*) for every $v \in \{v' \in \mathbb{R}^m_+ : v'_f = 1\}$[17]

$$\alpha^d_f(\alpha_{-f},v) \in \underset{\alpha_f\in\Delta(A_f)}{\arg\max} \sum_{g\in M} v_g\pi_g(\alpha_f,\alpha_{-f}),$$

and for every $s \in S$ and $\alpha_{-f} \in \times_{g\neq f}\Delta(A_g)$

(ii) (*disagreement payoffs derived from random dictatorship*)

$$d_{*f}(\alpha_{-f},s) = \sum_{i\in N_f(\delta_{*f})} \delta_{if}(s_{*f})u\left(\alpha^d_f(\alpha_{-f},\lambda_{i;f*}),\alpha_{-f},s\right),$$

[17]Notice that the choice function $\alpha^d_f(\alpha_{-f}, v)$ does not depend on the absolute size of a shareholder's stakes in the firms but only on her relative holdings v. This makes sense because $\arg\max_{\alpha_f\in\Delta(A_f)}\sum_{g\in M} v_g\pi_g(\alpha_f,\alpha_{-f})$ does not change if the objective function is multiplied by a positive constant. Also, notice that the choice function is not shareholder-specific. That is, all shareholders with the same relative holdings v choose the same strategy to be implemented by firm f in case of disagreement (if they are chosen by the lottery to make a decision). Of course, both of these properties are automatically satisfied when $\arg\max_{\alpha_f\in\Delta(A_f)}\sum_{g\in M} v_g\pi_g(\alpha_f,\alpha_{-f})$ is a singleton.

where $N_f(\delta_{*f}) \equiv \{i \in N : \delta_{if}(s_{*f}) > 0\}$ and $\lambda_{i;f*} \equiv s_{i*}/s_{if}$,

(iii) (*control exclusive to shareholders*) for every $i \in N$, $s_{if} = 0 \implies \delta_{if}(s_{*f}) = 0$.

RD disagreement payoffs have certain desirable properties. First, the disagreement payoffs are derived from a well-specified procedure. Second, they are feasible without the need for (free) disposal of profits. Third, through the probabilities δ with which different shareholders get to implement their most preferred strategy, the RD disagreement payoffs account for the relative power of shareholders.

Fourth, consider the case where A_f is a convex subset of a Euclidean space, and the portfolio profit of each firm f controlling shareholder is strictly concave in f's (pure) strategy a_f.[18] If firm f's controlling shareholders' preferences are not perfectly aligned,[19] then the shareholders have strict incentives to reach an agreement. Namely, by Jensen's inequality, every controlling shareholder will strictly prefer (to disagreement) that the firm implement the pure strategy that is the convex combination of the controlling shareholders' most-preferred strategies,[20] so the solution to the Nash bargaining problem is interior.

Last, while the NB mechanism can—much like the WAPP mechanism—be thought of as an as-if assumption, NB with RD disagreement payoffs (NBRD) also has connections to strategic foundations of Nash bargaining. For example, Howard (1992) shows that symmetric NBRD can be implemented as the unique perfect equilibrium outcome of a game.

D Equilibrium existence

Under NB, the equilibrium is a Nash equilibrium in Nash bargains. Particularly, the oligopoly game can be seen as a generalized game where a firm's strategy set depends on the other firms' strategies. Namely, when the other firms play α_{-f}, firm f can choose a strategy in $B_f(\alpha_{-f}, s)$, because it needs to make sure that each controlling shareholder achieves at least her disagreement payoff. Proposition 6 provides sufficient conditions for existence of a pure equilibrium of this generalized game.

Proposition 6. Fix an $s \in S$. If for every firm $f \in M$

(i) A_f is a non-empty, compact and convex subset of a Euclidean space,

(ii) $\pi_f(a)$ is continuous in a,

[18]Lemma 3 in the Appendix provides sufficient conditions for strict concavity in a homogeneous product Cournot market.

[19]That is, there exist distinct $i,j \in N$ such that $\delta_{if}(s_{*f}),\delta_{jf}(s_{*f}) > 0$ and $\alpha_f^d(\alpha_{-f},\lambda_{i;f*}) \neq \alpha_f^d(\alpha_{-f},\lambda_{j;f*})$, which are singletons and pure strategies by strict concavity. When firm f's controlling shareholders' preferences *are* perfectly aligned, then in case of disagreement, the strategy that is most preferred by all of them is chosen.

[20]That is, $u_i\left(\sum_{j\in N_f(\beta_{*f})} \delta_{jf}(s_{*f})\alpha_f^d(\alpha_{-f},\lambda_{j;f*}), \alpha_{-f}, s_{f*}\right) > d_{if}(\alpha_{-j},s)$ for every $i \in N_f(\beta_{*f})$.

(iii) for each $i \in N$, $d_{if}(a_{-f},s)$ is continuous in a_{-f},

(iv) $B_f^P(a_{-f})$ is lower hemicontinuous in a_{-f} over $a_{-f} \in \widetilde{A}_{-f}$,[21]

(v) $\pi_f(a_f,a_{-f})$ is concave in a_f for every $a_{-f} \in A_{-f}$,[22]

where $B_f^P(a_{-f}) \coloneqq \{a_f \in A_f : u_i(a_f,a_{-f},s_{i*}) \geq d_{if}(a_{-f},s)\ \forall i \in N_f(\beta_{*f})\}$ and $\widetilde{A} \coloneqq \{a \in A : a_f \in B_f^P(a_{-f})\ \forall f \in M\}$. Then, a pure Nash equilibrium in Nash bargains exists.

Lemma 2 provides conditions for assumption (iv) of Proposition 6 to hold.

Lemma 2. Fix an $s \in S$ and let condition (i) of Proposition 6 hold. For each firm $j \in M$ let the corporate control mechanism R_j be $\mathrm{NB}_{\beta_{*j},d_{*j}}$. $B_j^P(a_{-j})$ is lower hemicontinuous in $a_{-j} \in \widetilde{A}_{-j}$ if any of the following three conditions hold.

(i) For every $j \in M$, conditions (ii) and (v) of Proposition 6 hold, and for every $a_{-j} \in \widetilde{A}_{-j}$ there exists $a_j \in A_j$ such that $u(a_j,a_{-j},s) \gg d_{*j}(a_{-j},s)$.

(ii) For every $j \in M$, conditions (ii) and (iii) of Proposition 6 hold and for every $a_{-j} \in \widetilde{A}_{-j}$, $B_j^P(a_{-j},s) \subseteq \mathrm{cl}(\{a_j \in A_j : u(a_j,a_{-j},s) \gg d_{*j}(a_{-j},s)\})$.

(iii) For every $j \in M$, $\widetilde{A}_j \subset \mathbb{R}^{r_j}$ is an r_j-dimensional compact and convex polytope.

E Competitive effects of common ownership and policy implications

This section shows that WAPP and NB can give rise to significantly different theoretical predictions and policy implications. Specifically, I look at how market outcomes change as a shareholder varies the degree of diversification of a fixed number of shares across the industry.

Consider a homogeneous product Cournot duopoly ($m = 2$) with 3 shareholders ($n = 3$), linear inverse demand $P(Q) = \max\{10 - Q,0\}$ and symmetric linear cost functions $C_1(q_1) = q_1$, $C_2(q_2) = q_2$. Under both NBRD and WAPP, let control be proportional $\beta(s) = \gamma(s) = \delta(s) = s$, and the ownership structure be

$$s = \begin{bmatrix} s_{11} & 0.45 - s_{11} \\ 1 - s_{11} & 0 \\ 0 & 0.55 + s_{11} \end{bmatrix},$$

which is indexed by the shares s_{11} of shareholder 1 in firm 1.

The two firms are equally efficient and shareholder 1 (e.g., a large fund) can choose how to distribute her total holdings of 0.45 in the industry between the two firms.

[21] Lemma 2 in the Appendix provides sufficient conditions for condition (iv) to hold.
[22] Assumption (v) guarantees that the Nash product is quasi-concave in a_f.

Shareholders 2 and 3 are passive in that they are indifferent towards the capital they invest in the firms. The fund can buy shares of either firm at the same price and the rest of the capital is provided by shareholders 2 and 3. Define the normalized value $t \coloneqq (s_{11} - 0.225)/0.225 \in [-1,1]$ measuring what firm the fund's holdings are concentrated in. The closer t is to 0, the higher is the fund's diversification; for $t = 0$ the equilibrium is symmetric. As t increases shareholder 1's holdings become more concentrated in firm 1.

Think of a policy that limits the degree of common ownership a shareholder can have within the industry; it specifies some $\tau \in [0,1]$ and requires that $t \in [-1, -\tau] \cup [\tau,1]$. Figure 1 shows equilibrium results under NBRD and WAPP.

If the fund only cares to maximize its portfolio profit, then under WAPP it will choose t as close to 0 as possible. Thus, the price is decreasing in the restrictiveness τ of the policy. However, under NBRD the fund picks t as close as possible to either of the two peaks (in its portfolio profit) as possible, so that the price is first constant and then decreasing in τ. Therefore, a policy that is effective in increasing consumer welfare under WAPP may be ineffective under NBRD.[23]

Consider now an alternate scenario where the fund only cares to maximize its portfolio diversification, that is $\min |t|$, in order for example to mitigate risk or track an industry index. Then, under WAPP, the price is decreasing in τ. However, under NBRD, the price is first increasing and then decreasing in τ. Thus, a policy that is effective under WAPP may in fact harm consumer welfare under NBRD.

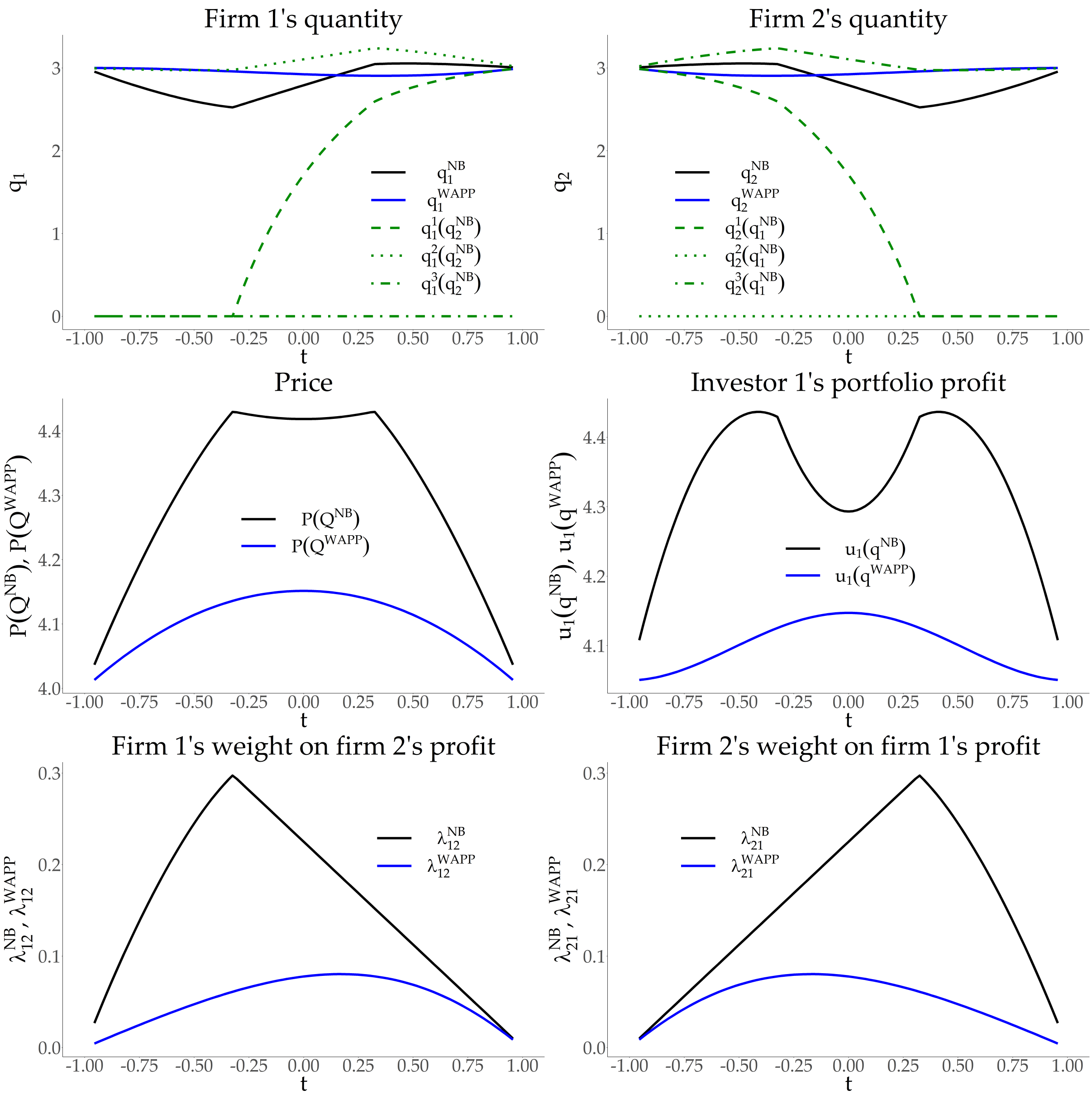
The differences in predictions between WAPP and NBRD are due to the differences (between the two mechanisms) in magnitudes of the various channels through which a change in t affects equilibrium outcomes. As t changes, both the fund's preferences and the division of power within each firm change.

Under WAPP, as t (*i.e.*, s_{11}) increases, the degree to which the fund wants firm 1 (resp. 2) to internalize firm 2's (resp. 1's) profits decreases (resp. increases), which tends to shift production towards firm 1. On the other hand, as t increases shareholder 2's control of firm 1 decreases, and shareholder 3's control of firm 2 increases, which tend to shift production towards firm 2. Under WAPP, around $t = 0$, the latter effects dominate, so that firm 2's quantity increases with t, while the quantity of firm 1 decreases making it unprofitable for the fund to pick $t \neq 0$. Also, firm 1's quantity increases faster than firm 2's quantity decreases with t (around $t = 0$), and the price has a global maximum at $t = 0$ under.

However, under NBRD, as t increases (around $t = 0$), production shifts towards firm 1, which is in the interest of the fund when $t > 0$. This makes it profitable for the fund to pick $t \neq 0$. Also, firm 1's quantity increases more slowly than firm 2's quantity decreases with t (around $t = 0$), so that the price has a local minimum at $t = 0$ under NBRD.

[23]Remember that consumer surplus is increasing in the total quantity (and thus decreasing in the price) in a homogeneous product market.

Figure 1: Equilibrium with a large fund and two undiversified passive shareholders for varying levels of diversification by the fund



Note: black lines represent equilibrium values under NBRD; blue ones under WAPP. Green lines show the most preferred quantity of each shareholder for each firm with the competitor's quantity taken as given (fixed at its equilibrium value). The bottom two panels plot λ_{12},λ_{21} (under WAPP) and $\widetilde{\lambda}_{12}$,$\widetilde{\lambda}_{21}$ (under NBRD).

Similarly, based on WAPP a consumer-welfare-maximizing regulator would want to block a trade that brings t from -0.25 to 0, even though this trade would increase consumer welfare under NBRD.

Last, notice that the graphs of control weights γ and $\tilde{\gamma}$ differ between WAPP and NBRD. These weights capture the extent to which changes in shareholder preferences (e.g., due to a stock trade) will be accommodated by each firm. Thus, the WAPP and NBRD models will give different predictions regarding stock trade effects.

F Application: homogeneous product Cournot oligopoly

This section characterizes the Nash-in-Nash equilibrium of a homogeneous product Cournot oligopoly and studies how changes in corporate control affect equilibrium outcomes.[24]

F.1 A Nash-in-Nash model of Cournot oligopoly with common ownership

There is a set N of n firms producing a homogeneous good. Each firm f chooses its production quantity q_f simultaneously with the other firms. Denote by $w_f \equiv q_f/Q$ firm f's market share of the total quantity $Q \coloneqq \sum_{g=1}^{n} q_g$. q_{-f} denotes the production profile of the firms other than f, and $Q_{-f} \coloneqq \sum_{g\neq f}^{n} q_g$. Firm f's production cost is given by the twice-differentiable function $C_f : \mathbb{R}_+ \to \mathbb{R}_+$ with $C_f'(q_f) > 0$ globally.

The twice-differentiable inverse demand function $P(Q)$ satisfies $P'(Q) < 0\ \forall Q \in \left[0,\overline{Q}\right)$, where $\overline{Q} \in (0, +\infty]$ is such that $P(Q) > 0 \iff Q \in \left[0,\overline{Q}\right)$. $\eta(Q) \coloneqq -P/(QP')$ denotes the elasticity of demand. Firm f's profit is given by $\pi_f(q) \coloneqq q_f P(Q) - C_f(q_f)$.

Define the following index of the weight firm f places on other firms' profits

$$\overline{\lambda}_j(q,s) \coloneqq \sum_{g\in M\setminus\{f\}} w_g \tilde{\lambda}_{fg}(q_{-f},s) \equiv \sum_{g\in M\setminus\{f\}} w_g \frac{\sum_{i\in N_f(\beta_{*f})} \tilde{\gamma}_{if}(q_{-f},s)s_{ig}}{\sum_{i\in N_f(\beta_{*f})} \tilde{\gamma}_{if}(q_{-f},s)s_{if}}.$$

Similarly, for each firm f and each shareholder i of firm f define $\overline{\lambda}_{i;f}(q,s_{i*}) \coloneqq \sum_{g\in M\setminus\{f\}} w_g \lambda_{i;jg}$, an index of the weight shareholder i wants firm f to place "on average" on other firms' profits.

Define also the bargaining-adjusted (i) Herfindahl-Hirschman Index (HHI) of market shares, (ii) MHHIΔ, and (iii) modified HHI, (iv) weighted average Lerner index LI, respectively given by

$$\mathrm{HHI}(q) \coloneqq \sum_{g\in M} w_g^2, \quad \mathrm{MHHI\Delta}(q,s) \coloneqq \sum_{g\in M} w_g \overline{\lambda}_g(q,s),$$
$$\mathrm{MHHI}(q,s) \coloneqq \mathrm{HHI}(q) + \mathrm{MHHI\Delta}(q,s), \quad \overline{\mathrm{LI}}(q) \coloneqq \sum_{g=1}^{m} w_g \frac{P(Q) - C_g'(q_g)}{P(Q)}.$$

[24]As seen in section 3, the analysis is also valid under WAPP.

F.2 The firm's problem in a homogeneous-product Cournot market

Lemma 3 provides conditions under which in a Cournot oligopoly a shareholder's portfolio profit is strictly concave in a firm's quantity.

Lemma 3. Fix a shareholder $i \in N$ and a firm $j \in M$. If for every quantity profile q such that $Q < \overline{Q}$ it holds that

$$E(Q) \sum_{k \in M} s_{ik} w_k < 1 + s_{ij} \left(1 - \frac{C''(q_j)}{P'(Q)}\right),$$

where $E(Q) := -P''(Q)Q/P'(Q)$ the (absolute value of the) elasticity of the slope of inverse demand, then for any q_{-j}, $u_i(q,s_{i*})$ is strictly concave in q_j for every q_j such that $Q < \overline{Q}$. A sufficient condition is

$$E(Q) < \frac{1 + s_{ij}}{\max_{k \in M} s_{ik}} \; \forall Q \in \left[0,\overline{Q}\right) \quad \text{and} \quad C''(q_j) \geq 0 \; \forall q_j.$$

Lemma 4 characterizes a firm's problem in a Cournot oligopoly.

Lemma 4. Assume that assumed there exists $\overline{q} > 0$ such that $P(q) < C_j(q)/q$ for every $q > \overline{q}$ and every firm $j \in M$. Fix a firm $j \in M$ and q_{-j} and let the corporate control mechanism R_j be $\mathrm{NB}_{\beta_{*j},d_{*j}}$. Assume that for every shareholder $i \in N$, $u_i(q,s_{i*})$ is strictly concave in q_j. Then, the following statements are true:

(i) $B_j^P(q_{-j},s) := \{q_j \in A_j : u\left(q_j,q_{-j},s\right) \geq d_{*j}(q_{-j},s)\}$ is a closed interval,

(ii) $R_j\left(q_{-j},s\right)$ is a singleton,

(iii) the Nash product is increasing (resp. decreasing) in q_j for $q_j \overset{(\text{resp.}>)}{<} R_j\left(q_{-j},s\right)$, and

(iv) if $\exists q_j$ such that $d_i(q_{-j},s) < u_i\left(q_j,q_{-j},s_{i*}\right)$ for every $i \in N_j(\beta_{*j})$, then $R_j\left(q{-}j,s\right)$ solves the FOC.

F.3 Nash-in-Nash equilibrium characterization

Let $\widetilde{S} \subseteq S$ be an open subset of S such that for every $s \in \widetilde{S}$, there is a unique and interior equilibrium q^* where $u\left(\mathrm{NB}_{\beta_{*f},d_{*f}}\left(q^*_{-f},s\right),q^*_{-f},s\right) \gg d_{*f}(q^*_{-f},s)$ for every firm $j \in M$. $q^* : \widetilde{S} \to \mathbb{R}^m_{++}$ returns this equilibrium as a function of s.[25] Similarly, write $Q^* \equiv \sum_{g \in M} q^*_g$, $w^*_f := q^*_f/Q^*$. To simplify notation, define also $\gamma^*_{if}(s) := \widetilde{\gamma}_{ij}(q^*_{-f}(s),s)$, $\lambda^*_{fg}(s) := \widetilde{\lambda}_{fg}(q^*_{-f}(s),s)$, $\overline{\lambda}^*_f(s) := \overline{\lambda}_f(q^*(s),s)$, $\overline{\lambda}^*_{i;f}(s) := \overline{\lambda}_{i;f}(q^*(s),s_{i*})$ for every shareholder $i \in N$ and pair of distinct firms $f,g \in M$. These functions give the equilibrium values of

[25] I will sometimes simply write q^* instead of $q^*(s)$.

the corresponding objects as functions of the ownership structure. $q^*(s)$ is then pinned down by the following FOCs:

$$f(q,s) := \left(\sum_{i\in N_1(\beta_{*1})} \gamma^*_{i1}(s)\frac{\partial u_i(q,s_{i*})}{\partial q_1} \quad \dots \quad \sum_{i\in N_m(\beta_{*m})} \gamma^*_{im}(s)\frac{\partial u_i(q,s_{i*})}{\partial q_m}\right)\Big|_{q=q^*(s)} = \mathbf{0}.$$

Denote the Jacobian of $f(q,s)$ (with respect to q) by $J(q,s)$. An interior, regular equilibrium is then defined as follows.

Definition 17. An equilibrium q^* is called interior and regular if (i) $q^* \gg \mathbf{0}$, (ii) for every firm $j \in M$, $d_{N_f(\beta_{*f})f}(q^*_{-f},s) \ll u_{N_f(\beta_{*f})}\left(q^*_f,q^*_{-f},s\right)$, and (iii) $J(q^*,s)$ is negative definite.

It is a maintained assumption that the equilbirium is interior and regular. Proposition 7 derives the equilibrium markup of each firm and the relationship between the weighted average Lerner index and the MHHI.

Proposition 7. In equilibrium for every firm $j \in M$ it holds that

$$\frac{P(Q^*) - C'_f(q^*_f)}{P(Q^*)} = \frac{w^*_f + \overline{\lambda}^*_f(s)}{\eta(Q^*)}.$$

The weighted average Lerner Index is $\overline{\mathrm{LI}}(q^*) = \mathrm{MHHI}(q^*,s)/\eta(Q^*)$.

F.4 Competitive effects of changes in corporate control

Consider an exogenous change in a shareholder's control power over a firm.

Definition 18. An exogenous increase (resp. decrease) in shareholder i's control over firm f at $s \in S \times \mathbb{R}^m_+$ is a change in the corporate control mechanism of firm f so that $\beta_{if}(s_{*f})$ changes infinitesimally by $d\beta_{if} >$ (resp. $<$) 0 with all else kept constant.[26]

Proposition 8 then studies the effects of a change in a shareholder's control over a firm.

Proposition 8. An exogenous increase (resp. decrease) in shareholder i's control over firm f causes firm f's quantity to change in the direction (resp. direction opposite to the one) preferred by shareholder i, that is

$$\mathrm{sgn}\left\{\frac{dq^*_f}{d\beta_{if}}\right\} = \mathrm{sgn}\left\{\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_f}\Bigg|_{q=q^*}\right\} = \mathrm{sgn}\left\{\overline{\lambda}^*_f(s) - \overline{\lambda}^*_{i;f}(s)\right\}.$$

Proposition 8 shows that if a firm is underproducing (resp. overproducing) relative to a shareholder's preferences and that shareholder's control over that firm increases, then the firms quantity will increase (resp. decrease). The proposition also provides

[26]For the entries of β_{*f} to still sum up to 1, the other entries clearly need to decrease. However, this is just a normalization that does not affect the analysis, so it is ignored. Also, notice that an exogenous increase (resp. decrease) in d_{if} will have the same qualitative effect as an increase (resp. decrease) in β_{if}.

an intuitive measure of whether the firm is under- or overproducing relative to the shareholder's preferences. It underproduces (resp. overproduces) if its (local) weighted average Edgeworth coefficient $\overline{\lambda}^*_f(s)$ is higher (resp. lower) than the shareholder's weighted Edgeworth coefficient.

A policy proposal by Posner et al. (2017) is to require institutional investors to be passive if they accumulate large amounts of stock in multiple competing firms. Such a policy can be understood as setting $\beta_{if} = 0$ for an investment fund i and every firm f. Provided that total quantity changes in the same direction as firm f's quantity, this policy will indeed increase consumer welfare if $\overline{\lambda}^*_{i;f}(s) > \overline{\lambda}^*_f(s)$ along a path where β_{if}'s go to 0 for every firm f.[27]

G Proofs of supplementary results

Proof of Proposition 6. The game can be seen as a generalized game where the strategy constraint correspondence is $B^P_j(a_{-j},s) := \{a_j \in A_j : u\,(a_j,a_{-j},s) \geq d_{*j}(a_{-j},s)\}$. The proof is composed of three steps.

Step 1: $B^P_j(a_{-j},s)$ is

(i) non-empty by property (i) of disagreement payoffs of NB mechanisms,

(ii) compact as a closed subset of a compact set (since u is continuous in a_j),

(iii) upper hemicontinuous in a_{-j}, as a closed-valued correspondence to a compact space (see, e.g., Corollary 9 in p.111, Aubin and Ekeland, 1984),

(iv) lower hemicontinuous in a_{-j} by assumption.

Also, the Nash product is continuous in a_j and a_{-j} given that u and d_{*j} are. It follows then by Berge's maximum theorem that $R_j\,(a_{-j},s)$ is an upper hemicontinuous, non-empty-valued and compact-valued correspondence.

Step 2: For any $i \in N$ and any $a_{-j} \in A_j$ we have that $u_i\left(\delta a_j + (1-\delta)a'_j,a_{-j},s_{i*}\right) - d_{ij}(a_{-j},s)$ is concave over $B^P_j(a_{-j},s)$. It follows that for any $i \in N_j(\beta_{*j})$ and any a_{-j}

$$\left(u_i\left(\delta a_j + (1-\delta)a'_j,a_{-j},s_{i*}\right) - d_{ij}(a_{-j},s)\right)^{\beta_{ij}(s_{*j})}$$

is concave (and thus log-concave) over $B^P_j(\alpha_{-j},s)$, since $a_j \mapsto u_i\left(\delta a_j + (1-\delta)a'_j,a_{-j},s_{i*}\right) - d_{ij}(a_{-j},s)$ is concave and $x \mapsto x^{\beta}_{ij}(s_{*j})$ is concave and increasing. Thus,

$$\prod_{i \in N_j(s)} \left(u_i\,(a_j,a_{-j},s_{i*}) - d_{ij}(a_{-j},s)\right)^{\beta_{ij}(s_{*j})}$$

[27] Under WAPP, the total quantity changes in the same direction as firm f's quantity if the game is aggregative and the slope of each firm's best response function is higher than -1 (see, e.g., Farrell and Shapiro, 1990; Vives, 1999). The game is aggregative if s is such that for every firm $j \in M$, $\lambda_{fg}(s) = \lambda_{fh}(s)$ for every pair of firms $g,h \in M \setminus \{f\}$.

is log-concave over $B_j^P(a_{-j},s)$ as a product of log-concave functions (and thus also quasi-concave in a_j for every a_{-j}). The product is also continuous in a_j and a_{-j}, and given also that $B_j^P(a_{-j},s)$ is convex for any $a_{-j} \in A_{-j}$, it follows that $R_j\,(a_{-j},s)$ is convex-valued.

Step 3: $G(a) \coloneqq \times_{j \in M} R_j(a_{-j},s)$ is an upper hemicontinuous, non-empty-, compact- and convex-valued correspondence since R_j is for each $j \in M$. By Kakutani's fixed point theorem, G admits a fixed point, which is an equilibrium. **Q.E.D.**

Proof of Lemma 2. Part (i) follows from Proposition 4.2 in Dutang (2013), which is an application of Theorem 5.9 in Rockafellar and Wets (1997). Part (ii) follows from Proposition 4.3 in Dutang (2013); see also Theorem 13 of Hogan (1973). Part (iii) follows from Corollary 2 in Maćkowiak (2006). A similar result is also given in Claim 2 of Banks and Duggan (2004). **Q.E.D.**

Proof of Lemma 3. The derivative of $u_i\,(q_j,q_{-j},s_{i*})$ with respect to q_j is given by

$$\frac{\partial u_i\,(q_j,q_{-j},s_{i*})}{\partial q_j} = s_{ij}\,(P(Q) - C'(q_j)) + P'(Q) \sum_{k \in M} s_{ik} q_k,$$

and the second derivative by

$$\begin{aligned}\frac{\partial^2 u_i\,(q_j,q_{-j},s_{i*})}{\partial q_j^2} &= (1 + s_{ij})P'(Q) - s_{ij}C''(q_j) + P''(Q) \sum_{k \in M} s_{ik} q_k \\ &= P'(Q)\left[1 + s_{ij}\left(1 - \frac{C''(q_j)}{P'(Q)}\right) - E(Q) \sum_{k \in M} s_{ik} w_k\right],\end{aligned}$$

and the result follows. **Q.E.D.**

Proof of Lemma 4. Since for $q_j > \overline{q}$ profit becomes negative, we can constrain each firm to choose quantity $q_j \in [0,\overline{q}]$. From continuity of u_i in q_j and the definition of B_j^P it follows then that B_j^P is compact. Especially, given strict concavity of u_i in q_j for every i, it follows that B_j^P is convex, thus a closed interval. We distinguish the following two cases:

Case 1: Given that u_i is strictly concave in q_j for every i (so u_i can be equal to d_{ij} for at most 2 values of q_j in B_j^P), the only way that $\forall q_j \in B_j^P$ there exists $i \in N$ such that $d_{ij}(q_{-j},s) = u_i\,(q_j,q_{-j},s_{i*})$ is for B_j^P to be a singleton. By continuity of u_i in q_j, this means that $d_{ij}(q_{-j},s)$ is equal to $\max_{q_j} u_i\,(q_j,q_{-j},s_{i*})$ for some $i \in N$, and the relevant results follow.

Case 2: If $\exists q_j \in B_j^P$ such that $d_{*j}(q_{-j},s) \ll u\,(q_j,q_{-j},s)$, we have that for every $i \in N$

and every $q_j \in B_j^P(q_{-j},s)$

$$\begin{aligned}
&\frac{\partial^2 \left(u_i\left(q_j,q_{-j},s_{i*}\right) - d_{ij}(q_{-j},s)\right)^{\beta_{ij}(s_{*j})}}{\partial q_j^2} \\
&= - \frac{\beta_{ij}(s_{*j})\left(1-\beta_{ij}(s_{*j})\right)}{\left(u_i\left(q_j,q_{-j},s_{i*}\right) - d_{ij}(q_{-j},s)\right)^{2-\beta_{ij}(s_{*j})}} \left(\frac{\partial u_i\left(q_j,q_{-j},s_{i*}\right)}{\partial q_j}\right)^2 \\
&\quad + \frac{\beta_{ij}(s_{*j})}{\left(u_i\left(q_j,q_{-j},s_{i*}\right) - d_{ij}(q_{-j},s)\right)^{1-\beta_{ij}(s_{*j})}} \frac{\partial^2 u_i\left(q_j,q_{-j},s_{i*}\right)}{\partial q_j^2} < 0,
\end{aligned}$$

by strict concavity of u_i in q_j. Also, for every i, $\left(u_i\left(q_j,q_{-j},s_{i*}\right) - d_{ij}(q_{-j},s)\right)^{\beta_{ij}(s_{*j})}$ is non-negative and not identically equal to zero over B_j^P. The results then follow from Theorem 4 in Kantrowitz and Neumann (2005). **Q.E.D.**

Proof of Proposition 7. The FOCs in equilibrium give:

$$P(Q^*) - C_j'(q_j^*) + P'(Q^*)\left[q_j^* + \sum_{k \in M\setminus\{j\}} \lambda_{jk}^*(s) q_k^*\right] = 0,$$

and the result follows. **Q.E.D.**

Proof of Proposition 8. The partial derivative of $f(q,s)$ with respect to β_{ij} is

$$\begin{aligned}
\frac{\partial f(q,s)}{\partial \beta_{ij}} &= \left[\frac{\gamma_{ij}^*}{\beta_{ij}} \frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j} - \frac{1}{u_i - d_{ij}} \frac{1}{\sum_{h \in N_j(\beta_{*j})} \frac{\beta_{hj}}{u_h - d_{hj}}} \sum_{t \in N_j(\beta_{*j})} \gamma_{tj}^* \frac{\partial u_t\left(q,s_{t*}\right)}{\partial q_j}\right] \cdot \mathbf{e}_j \\
&= \frac{\gamma_{ij}^*}{\beta_{ij}} \frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j} \cdot \mathbf{e}_j,
\end{aligned}$$

where $\mathbf{e}_j$ the m-dimensional standard unit vector with 1 in its j-th dimension. It follows by the Implicit Function Theorem that

$$\begin{aligned}
\begin{pmatrix} \frac{dq_1^*}{d\beta_{ij}} \\ \frac{dq_2^*}{d\beta_{ij}} \\ \vdots \\ \frac{dq_m^*}{d\beta_{ij}} \end{pmatrix} &= -J^{-1}(q^*,s) \left.\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j}\right|_{q=q^*} \cdot \mathbf{e}_j = -\left(\det\left(J\right)\right)^{-1} \frac{\partial u_i}{\partial q_j} \cdot \operatorname{adj}\left(J\right) \mathbf{e}_j \\
&= -\left(\det\left(J\right)\right)^{-1} \frac{\partial u_i}{\partial q_j} \cdot \begin{pmatrix} (-1)^{1+j} \det\left(J_{-j-1}\right) \\ (-1)^{2+j} \det\left(J_{-j-2}\right) \\ \vdots \\ (-1)^{m+j} \det\left(J_{-j-m}\right) \end{pmatrix},
\end{aligned}$$

where the second equality follows from the Laplace expansion, $\operatorname{adj}\left(J\right)$ is the adjugate or classical adjoint of J, and J_{-j-k} is the J matrix with the j-th row and k-th column removed. Since J is negative definite

$$\operatorname{sgn}\{\det(J)\} = -\operatorname{sgn}\{\det(J_{-j-j})\} = \operatorname{sgn}\{(-1)^m\},$$

$$\text{so that} \quad \operatorname{sgn}\left\{\frac{dq_j^*}{d\beta_{ij}}\right\} = \operatorname{sgn}\left\{(-1)^{2j}\frac{\partial u_i}{\partial q_j}\right\} = \operatorname{sgn}\left\{\left.\frac{\partial u_i\,(q,s_{i*})}{\partial q_j}\right|_{q=q^*}\right\},$$

where

$$\left.\frac{\partial u_i\,(q,s_{i*})}{\partial q_j}\right|_{q=q^*} = \sum_{h=1}^{m} s_{ih}\left.\frac{\partial \pi_h(q,s_{i*})}{\partial q_j}\right|_{q=q^*} = P(Q^*)\left[s_{ij}\frac{P(Q^*)-C_j'(q_j^*)}{P(Q^*)} - \frac{\sum_{h=1}^m s_{ih}w_h^*}{\eta(Q^*)}\right]$$
$$= -Q^*P'(Q^*)\left[s_{ij}\left(w_j^*+\overline{\lambda}_j\right) - \sum_{h=1}^{m} s_{ih}w_h^*\right] = -Q^*P'(Q^*)s_{ij}\left(\overline{\lambda}_j^* - \overline{\lambda}_{i;j}^*\right),$$

and the result follows. **Q.E.D.**

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